DSC 140B - Homework 01

Due: Wednesday, April 12

Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. Unless otherwise noted by the problem's instructions, show your work or provide some justification for your answer. Homeworks are due via Gradescope at 11:59 PM.

Problem 1.

Suppose that in a group of 1000 people, 600 currently live in California and 400 currently live in Texas. In any given year, 5% of the people living in California move to Texas, and 3% of the people living in Texas move to California. You may assume that the people do not move to any other states.

We can represent the current number of people living in California and Texas with a *population vector*:

$$\vec{p} = (\#\text{ in California}, \#\text{ in Texas})^T.$$

The initial situation described above is represented by the population vector $(600, 400)^T$.

a) After one year, how many people will be living in Calfornia and Texas? What about after two years?

Do this problem by hand, showing your calculations. State your answers in the form of a population vector. It is OK for your results to be decimals (*don't* round them to the nearest integer).

Solution: Let C_1 and T_1 be the *new* populations of California and Texas after one year (what we're trying to calculate). Let C_0 and T_0 be the current populations.

We have, after one year:

$$\begin{aligned} C_1 &= C_0 + .03 \times T_0 - .05 \times C_0 \\ &= 600 + .03 \times 400 - .05 \times 600 \\ &= 600 + 12 - 30 \\ &= 582 \end{aligned}$$

$$\begin{aligned} T_1 &= T_0 + .05 \times C_0 - .03 \times T_0 \\ &= 400 + .05 \times 600 - .03 \times 400 \\ &= 400 + 30 - 12 \\ &= 418 \end{aligned}$$

So, after one year, the population vector is $(582, 418)^T$.

We use these updated numbers to calculate the populations after an additional year:

$$\begin{aligned} C_2 &= C_1 + .03 \times T_1 - .05 \times C_1 \\ &= 582 + .03 \times 418 - .05 \times 582 \\ &\approx 582 + 12.54 - 29.1 \\ &\approx 565.44 \end{aligned}$$

$$\begin{aligned} T_2 &= T_1 + .05 \times C_1 - .03 \times T_1 \\ &= 418 + .05 \times 582 - .03 \times 418 \\ &= 418 + 29.1 - 12.54 \\ &= 434.56 \end{aligned}$$

So, after two years, the population vector is $(565.44, 434.56)^T$.

b) Let $\vec{f}(\vec{p})$ be the transformation which takes in a current population vector, $\vec{p} = (c, t)^T$, and returns the population vector after one year has passed.

Write the formula of the transformation in coordinate form with respect to the standard basis. That is, fill in:

$$\vec{f}(\vec{p}) = (\ldots, \quad \ldots)^T.$$

Example: consider the transformation $\vec{g}$ which doubles the population of California each year, and triples population of Texas. Written in coordinate form, that transformation has the formula $\vec{g}(\vec{p}) = (2c, 3t)^T$.

Hint: your answer should have the form:

$$\vec{f}(\vec{p}) = (\alpha_1 c + \alpha_2 t, \quad \alpha_3 c + \alpha_4 t)^T,$$

where $\alpha_1, \ldots, \alpha_4$ are real constants that you should provide.

Solution:

$$\vec{f}(\vec{p}) = (c + .03t - .05c, \quad t + .05c - .03t)^T$$

Or, simplified:

$$\vec{f}(\vec{p}) = (.95c + .03t, \quad .05c + .97t)^T$$

c) Prove that the transformation $\vec{f}(\vec{p})$ you derived above is a *linear* transformation by showing that it satisfies the definition. That is, show that for any vector $\vec{u} = (c_1, t_1)^T$ and $\vec{v} = (c_2, t_2)^T$, and scalars α, β:

$$\vec{f}(\alpha\vec{u} + \beta\vec{v}) = \alpha\vec{f}(\vec{u}) + \beta\vec{f}(\vec{v})$$

Solution: Let $\vec{u} = (c_1, t_1)^T$ and $\vec{v} = (c_2, t_2)^T$. Then:

$$\begin{aligned} \vec{f}(\alpha\vec{u} + \beta\vec{v}) &= \vec{f}(\alpha(c_1, t_1)^T + \beta(c_2, t_2)^T) \\ &= \vec{f}((\alpha c_1 + \beta c_2, \alpha t_1 + \beta t_2)^T) \\ &= \begin{pmatrix} .95(\alpha c_1 + \beta c_2) + .03(\alpha t_1 + \beta t_2) \\ .05(\alpha c_1 + \beta c_2) + .97(\alpha t_1 + \beta t_2) \end{pmatrix} \\ &= \begin{pmatrix} \alpha(.95c_1 + .03t_1) + \beta(.95c_2 + .03t_2) \\ \alpha(.05c_1 + .03t_1) + \beta(.03c_2 + .03t_2) \end{pmatrix} \\ &= \alpha \underbrace{\begin{pmatrix} .95c_1 + .03t_1 \\ .05c_1 + .03t_1 \end{pmatrix}}_{\vec{f}(\vec{u})} + \beta \underbrace{\begin{pmatrix} .95c_2 + .03t_2 \\ .03c_2 + .03t_2 \end{pmatrix}}_{\vec{f}(\vec{v})} \\ &= \alpha\vec{f}(\vec{u}) + \beta\vec{f}(\vec{v}) \end{aligned}$$

d) In 50 years, how many people will live in California, and how many will live in Texas? That is, what is the population vector $\vec{p}$ after $\vec{f}$ is applied 50 times? Your answer can include decimal numbers.

You (probably) do not want to carry out these calculations by hand. Instead, implement it in code (attaching a screenshot to show your work).

Solution: Approximately: $(378.47, \quad 621.52)^T$.

To find this, we write a Python function, `f(x)`, and run it in a for-loop, 50 times.

```
p_ct = .05
p_tc = .03

def f(x):
    c, t = x
    c_new = c - p_ct * c + p_tc * t
    t_new = t - p_tc * t + p_ct * c
    return np.array([c_new, t_new])

x = np.array([600, 400])
for i in range(500):
    x = f(x)
    print(x)
```

e) You might wonder: if this process is allowed to continue, will it ever *converge*? That is, will there ever be a time where the populations in California and Texas do not change from year to year?

To answer this, repeat the previous question, but increase the number of iterations until the population vector does not change. Report as your answer this population vector.

Hint: your answer should contain nice, whole numbers which add to 1000.

Solution: The "steady-state" population is $(375, 625)^T$.

We find this by running the for-loop in the code for the last part for a few hundred additional iterations.

f) Let $\vec{u}^{(1)}$ be the vector you found in the previous part. What you saw above is that $\vec{f}(\vec{u}^{(1)}) = \vec{u}^{(1)}$. In the language of linear algebra, $\vec{u}^{(1)}$ is an *eigenvector* of $\vec{f}$ with eigenvalue 1, since $\vec{f}(\vec{u}^{(1)}) = 1 \cdot \vec{u}^{(1)}$.

It can be shown that another eigenvector of $\vec{f}$ is $\vec{u}^{(2)} = (1, -1)^T$, and that $\vec{f}(\vec{u}^{(2)}) = 0.92\vec{u}^{(2)}$. In the language of linear algebra, the eigenvalue associated with $\vec{u}^{(2)}$ is 0.92.

It is often useful to use the eigenvectors of a linear transformation as basis vectors. In some situations, the eigenvectors are guaranteed to be orthogonal, although that is not the case here. However, we can still use the above eigenvectors as a basis, but it will not be an *orthonormal basis*.

Write the vector $\vec{p} = (600, 400)^T$ as a coordinate vector in the basis $\mathcal{U} = \{\vec{u}^{(1)}, \vec{u}^{(2)}\}$. That is, find $[\vec{p}]_{\mathcal{U}}$.

Hint: you cannot use the approach described in class, where we found the coordinates by computing $\vec{p} \cdot \vec{u}^{(1)}$ and $\vec{p} \cdot \vec{u}^{(2)}$. That approach only works in the case of an *orthonormal* basis. Instead, recognize that we want to find α and β so that $\vec{p} = \alpha\vec{u}^{(1)} + \beta\vec{u}^{(2)}$. This amounts to solving the system of two equations:

$$\underbrace{\begin{pmatrix} 600 \\ 400 \end{pmatrix}}_{\vec{p}} = \alpha \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{\vec{u}^{(1)}} + \beta \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\vec{u}^{(2)}},$$

where a and b are the coordinates of $\vec{u}^{(1)}$ that you found in the last problem (i.e., they are whole numbers which add to 1000).

Solution: We wish to solve the system of two equations:

$$\begin{pmatrix} 600 \\ 400 \end{pmatrix} = \alpha \begin{pmatrix} 375 \\ 625 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Or, written in a form that might be more familiar:

$$600 = 375\alpha + \beta$$
$$400 = 625\alpha - \beta$$

Adding the two equations, we get $1000 = 1000\alpha$, which implies $\alpha = 1$. Plugging this into one of the other equations (say, the second), we get $400 = 625 - \beta$, and so $\beta = 225$.

That is, $[\vec{p}]_{\mathcal{U}} = (1, 225)^T$.

g) Let $[\vec{x}]_{\mathcal{U}} = (x_1, x_2)^T$ be a population vector with respect to the basis $\mathcal{U}$. Write the formula for $\vec{f}(\vec{x})$ with respect to the basis $\mathcal{U}$. That is, what is $[\vec{f}(\vec{x})]_{\mathcal{U}}$?

Hint: $\vec{f}$ is linear, so $\vec{f}(\alpha\vec{u}^{(1)} + \beta\vec{u}^{(2)}) = \alpha\vec{f}(\vec{u}^{(1)}) + \beta\vec{f}(\vec{u}^{(2)})$. We already know what $\vec{f}(\vec{u}^{(1)})$ and $\vec{f}(\vec{u}^{(2)})$ are from above.

Solution: $[\vec{x}]_{\mathcal{U}} = (x_1, x_2)^T$ means that $\vec{x} = x_1\vec{u}^{(1)} + x_2\vec{u}^{(2)}$.

So:

$$\vec{f}(\vec{x}) = \vec{f}(x_1\vec{u}^{(1)} + x_2\vec{u}^{(2)})$$

And, because f is linear:

$$= x_1\vec{f}(\vec{u}^{(1)}) + x_2\vec{f}(\vec{u}^{(2)})$$

From above, we know that $\vec{f}(\vec{u}^{(1)}) = \vec{u}^{(1)}$ and $\vec{f}(\vec{u}^{(2)}) = 0.92\vec{u}^{(2)}$.

$$= x_1\vec{u}^{(1)} + 0.92x_2\vec{u}^{(2)}$$

So $[\vec{f}(\vec{x})]_{\mathcal{U}} = (1x_1, 0.92x_2)^T$.

h) Suppose $[\vec{x}]_{\mathcal{U}} = (x_1, x_2)^T$ is current population vector, expressed with respect to the basis $\mathcal{U}$. Write a formula for the population vector after k years have passed, also expressed in the basis $\mathcal{U}$.

Hint: your formula should involve x_1, x_2, k and some constants.

Your formula should be pretty simple – this was enabled by using the eigenvectors as our basis. The formula written in the standard basis is not nearly as simple!

Solution: After one year, the population vector is $(x_1, 0.92x_2)^T$; we found this in the last part. After two years, the population vector is:

$$\vec{f}((x_1, 0.92)^T) = (x_1, 0.92 \times 0.92x_2)^T = (x_1, 0.92^2x_2)^T$$

After k years, it will be $(x_1, 0.92^k \cdot x_2)^T$.

i) Suppose the current population vector, expressed in the basis $\mathcal{U}$, is $(1, 225)^T$. What will the population vector be in 50 years, expressed in terms of the basis $\mathcal{U}$?

Solution: From the previous part, after k years the population vector will be $(x_1, 0.92^k x_2)^T = (1, 0.92^k \times 225)^T$. So, after 50 years, the population vector will be:

$$\begin{aligned}(1, 0.92^{50} \times 225)^T &= (1, .015 \times 225)^T \\ &= (1, 3.375)^T\end{aligned}$$

j) Express the vector your found in the last part as a coordinate vector in the standard basis.

Hint: your result should be familiar as an answer to a previous part. However, it might not be *exactly* the same due to some roundoff error – you presumably calculated the other answer on a computer with finite numerical precision.

Solution: In the last problem, we saw that the population vector in the basis $\mathcal{U}$ after 50 years will be $(1, 3.375)^T$. That is, $\vec{p} = \vec{u}^{(1)} + 3.375\vec{u}^{(2)}$.

Since in the standard basis, $\vec{u}^{(1)} = (375, 625)^T$ and $\vec{u}^{(2)} = (1, -1)^T$, we have:

$$\begin{aligned}\vec{p} &= 1\vec{u}^{(1)} + 3.375 \\ &= \begin{pmatrix} 375 \\ 625 \end{pmatrix} + 3.375 \times \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 375 + 3.375 \\ 625 - 3.375 \end{pmatrix} \\ &= \begin{pmatrix} 378.375 \\ 621.625 \end{pmatrix}\end{aligned}$$