DSC 190 - Homework 07

Due: Wednesday, May 24

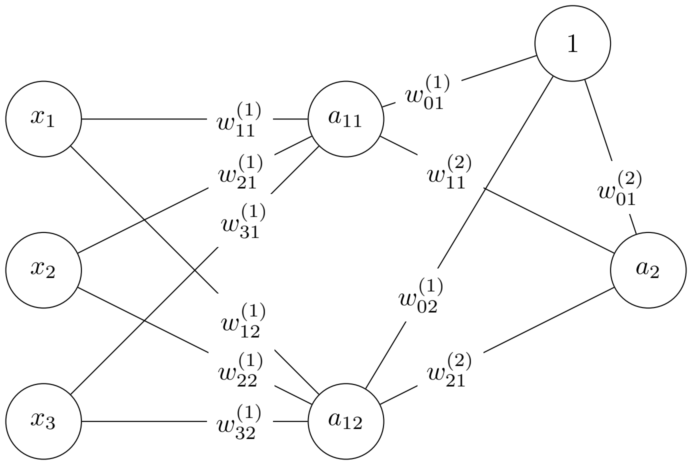
Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. Unless otherwise noted by the problem's instructions, show your work or provide some justification for your answer. Homeworks are due via Gradescope at 11:59 PM.

Problem 1.

In lecture it was said that a neural network with linear activation functions is a linear prediction function, meaning that its decision boundary will also be linear. If we wish to have a non-linear decision boundary, we must introduce non-linearities with, for instance, non-linear activation functions.

In this problem, we'll see concretely that a neural network with linear activations is again linear.

Consider the neural network shown below.



The inputs x_1, x_2, and x_3 are numbers. Each $w_{ij}^{(k)}$ is a scalar weight. a_{ij} denotes the output of a neuron. Remember that when linear activations are used, the output of a neuron is simply the weighted sum of its inputs. So for instance:

$$a_{11} = w_{11}^{(1)} x_1 + w_{21}^{(1)} x_2 + w_{31}^{(1)} x_3 + w_{01}^{(1)}$$

The node labeled 1 is the bias input. a_2 is the output of the neural network overall.

a) Write the output of the network, a_2, as an expression involving only the inputs x_1, x_2, x_3 and the weights, $w_{ij}^{(k)}$. a_i should not appear in your expression.

Solution: We have

$$a_{11} = w_{11}^{(1)} x_1 + w_{21}^{(1)} x_2 + w_{31}^{(1)} x_3 + w_{01}^{(1)}$$
$$a_{12} = w_{12}^{(1)} x_1 + w_{22}^{(1)} x_2 + w_{32}^{(1)} x_3 + w_{02}^{(1)}$$

Therefore:

$$\begin{aligned}
a_2 &= w_{11}^{(2)} a_{11} + w_{21}^{(2)} a_{12} + w_{01}^{(2)} \\
&= w_{11}^{(2)} \left(w_{11}^{(1)} x_1 + w_{21}^{(1)} x_2 + w_{31}^{(1)} x_3 + w_{01}^{(1)} \right) \\
&\quad + w_{21}^{(2)} \left(w_{12}^{(1)} x_1 + w_{22}^{(1)} x_2 + w_{32}^{(1)} x_3 + w_{02}^{(1)} \right) \\
&\quad + w_{01}^{(2)}
\end{aligned}$$

b) Show that the output of the network can be written

$$a_2 = w_0 + w_1 x_1 + w_2 x_2 + w_3 x_3,$$

where $w_0, w_1, w_2,$ and w_3 are scalars that depend only on the weights in the original network. By showing this, you're proving that the network above is equivalent to a much simpler linear model.

Solution: Starting from the result of the last subproblem:

$$\begin{aligned}
a_2 &= w_{11}^{(2)} \left(w_{11}^{(1)} x_1 + w_{21}^{(1)} x_2 + w_{31}^{(1)} x_3 + w_{01}^{(1)} \right) \\
&\quad + w_{21}^{(2)} \left(w_{12}^{(1)} x_1 + w_{22}^{(1)} x_2 + w_{32}^{(1)} x_3 + w_{02}^{(1)} \right) \\
&\quad + w_{01}^{(2)}
\end{aligned}$$

Grouping the terms involving x_1, x_2, and x_3 separately:

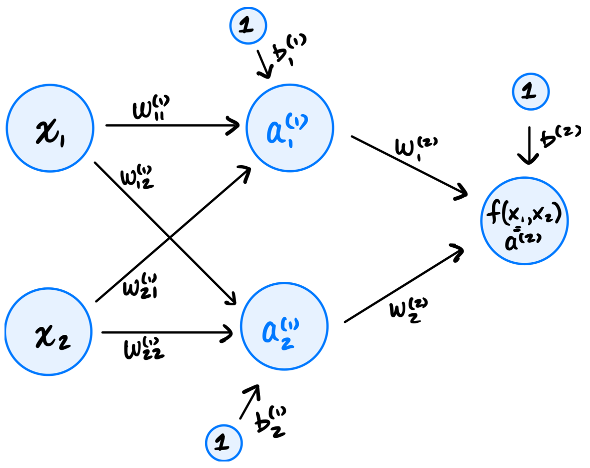
$$\begin{aligned}
&= \left(w_{11}^{(2)} w_{11}^{(1)} + w_{21}^{(2)} w_{12}^{(1)} \right) x_1 \\
&\quad + \left(w_{11}^{(2)} w_{21}^{(1)} + w_{21}^{(2)} w_{22}^{(1)} \right) x_2 \\
&\quad + \left(w_{11}^{(2)} w_{31}^{(1)} + w_{21}^{(2)} w_{32}^{(1)} \right) x_3 \\
&\quad + w_{01}^{(2)} + w_{11}^{(2)} w_{01}^{(1)} + w_{21}^{(2)} w_{02}^{(1)} \\
&= w_0 + w_1 x_1 + w_2 x_2 + w_3 x_3
\end{aligned}$$

where

$$\begin{aligned}
w_1 &= w_{11}^{(2)} w_{11}^{(1)} + w_{21}^{(2)} w_{12}^{(1)} \\
w_2 &= w_{11}^{(2)} w_{21}^{(1)} + w_{21}^{(2)} w_{22}^{(1)} \\
w_3 &= w_{11}^{(2)} w_{31}^{(1)} + w_{21}^{(2)} w_{32}^{(1)} \\
w_0 &= w_{01}^{(2)} + w_{11}^{(2)} w_{01}^{(1)} + w_{21}^{(2)} w_{02}^{(1)}
\end{aligned}$$

Problem 2.

Consider the network below:



Suppose

$$W^{(1)} = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix}$$
$$\vec{b}^{(1)} = (0, 0)^T$$

and assume that the hidden layer uses the ReLU as its activation function.

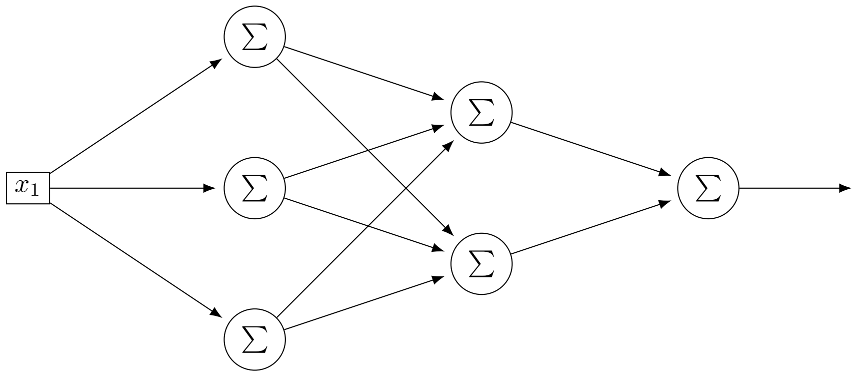
As discussed in lecture, the first layer of this neural network can be viewed as mapping an input point in $\mathbb{R}^2$ to a new representation, also in $\mathbb{R}^2$.

Suppose $\vec{x} = (2, 1)^T$. What is the new representation of $\vec{x}$ produced by this network?

Solution: The new representation is $(0, 8)^T$.

Problem 3.

Consider the neural network architecture shown below:



In all parts of this problem, assume that the network's parameters are:

$$W^{(1)} = \begin{pmatrix} -3 & -5 & 4 \end{pmatrix} \qquad W^{(2)} = \begin{pmatrix} 2 & -3 \\ -5 & 5 \\ -3 & 0 \end{pmatrix} \qquad W^{(3)} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$$

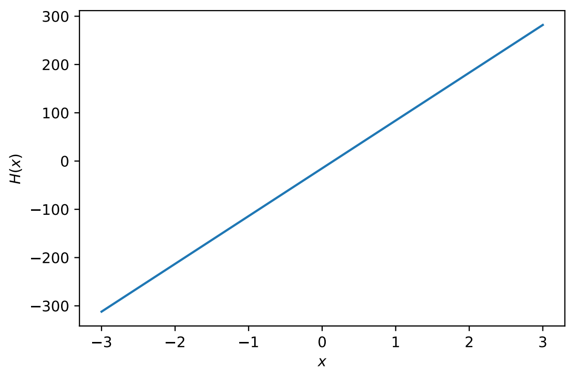
$$\vec{b}^{(1)} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \qquad \vec{b}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \vec{b}^{(3)} = \begin{pmatrix} -3 \end{pmatrix}$$

Note that this network is a function $H : \mathbb{R} \to \mathbb{R}$, so we can easily plot it. In the parts below, you will plot the network for a range of inputs. Your plots may not necessarily be *interesting*, but they will give you a sense of how different choices of activation function affect the type of function the neural network computes.

Suggestion: create a Python function `network(x, activation)` which takes in two things: a number `x` and an `activation` function, and computes the output of the network on x using that activation function (that is, it computes $H(x)$). You can then use that code for all parts of this problem.

a) Assume that all activation functions are linear. Plot $H(x)$ in the range $x \in [-3, 3]$. Show your code.

Solution:

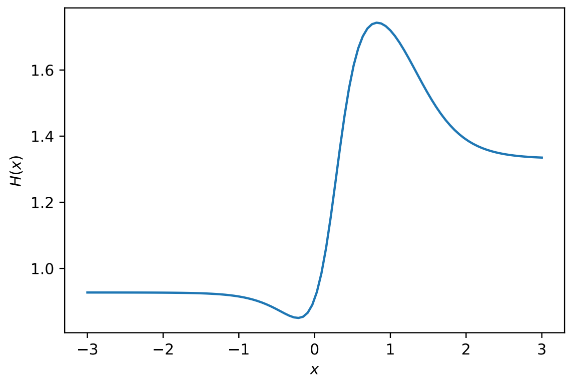


This was generated by the following code, which was also used to generate the images in the other subproblems.

```python
"""This code isn't the most *efficient* way to implement a neural
network, but it is maybe the simplest."""

import numpy as np
import matplotlib.pyplot as plot


def linear(x):
    return x

def sigmoid(x):
    return 1/(1 + np.exp(x))

def relu(x):
    return np.maximum(0, x)

def network(x, activation=linear):
    x = np.array([x])

    W_1 = np.array([-3, -5, 4])[:,None]
    b_1 = np.array([3, 2, -1])

    W_2 = np.array([
        [2, -3],
        [-5, 5],
        [-3, 0]
    ])
    b_2 = np.array([1, 2])

    W_3 = np.array([5, -4])
    b_3 = np.array([-3])

    def H_1(z):
        return activation(W_1 @ z + b_1)

    def H_2(z):
        return activation(W_2.T @ z + b_2)

    def H_3(z):
        # output node uses linear activation
        return W_3 @ z + b_3

    return H_3(H_2(H_1(x)))[0]

def H_linear(x):
    return network(x, activation=linear)

def H_relu(x):
    return network(x, activation=relu)

def H_sigmoid(x):
    return network(x, activation=sigmoid)

def plot(H):
    xx = np.linspace(-3, 3, 100)
    yy = [H(x) for x in xx]
    plt.plot(xx, yy)

    plt.xlabel('$x$')
    plt.ylabel('$H(x)$')
```

b) Assume that all hidden nodes have **sigmoid** activation and that the output node has linear activation. Plot $H(x)$ in the range $x \in [-3, 3]$. Show your code.

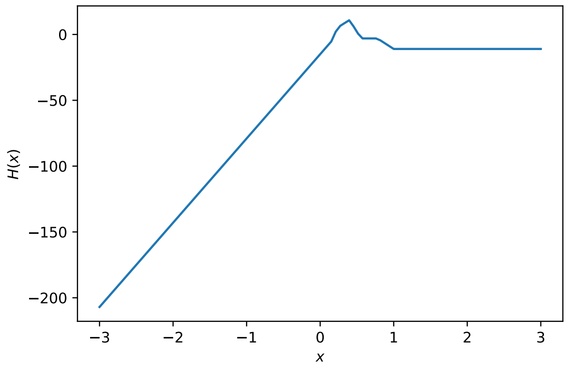
Solution:



c) Assume that all hidden nodes have **ReLU** activation and that the output node has linear activation.

Plot $H(x)$ in the range $x \in [-3, 3]$. Show your code.

Solution:



0
−50
−100
−150
−200
H(x)
−3
−2
−1
0
1
2
3
x