DSC 140B
Representation Learning

Lecture 03 | Part 1

Functions of a Vector

Functions of a Vector

- In ML, we often work with functions of a vector: $f : \mathbb{R}^d \to \mathbb{R}^{d'}$.
- Example: a prediction function, $H(\vec{x})$.
- Functions of a vector can return:
  - a number: $f : \mathbb{R}^d \to \mathbb{R}^1$
  - a vector $\vec{f} : \mathbb{R}^d \to \mathbb{R}^{d'}$
  - something else?

Transformations

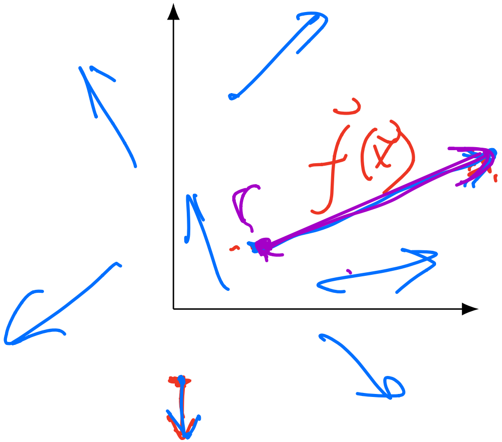
- A **transformation** $\vec{f}$ is a function that takes in a vector, and returns a vector *of the same dimensionality*.
- That is, $\vec{f} : \mathbb{R}^d \to \mathbb{R}^d$.

Visualizing Transformations

- A transformation is a **vector field**.
  - Assigns a vector to each point in space.
  - Example: $\vec{f}(\vec{x}) = (3x_1, x_2)^T$

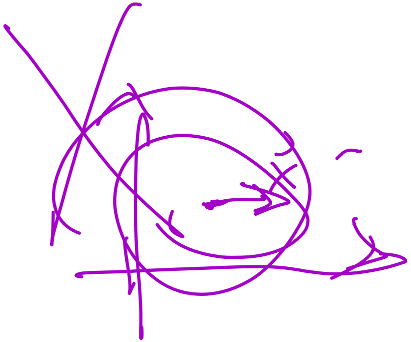
$\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{f}(\vec{x}) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$



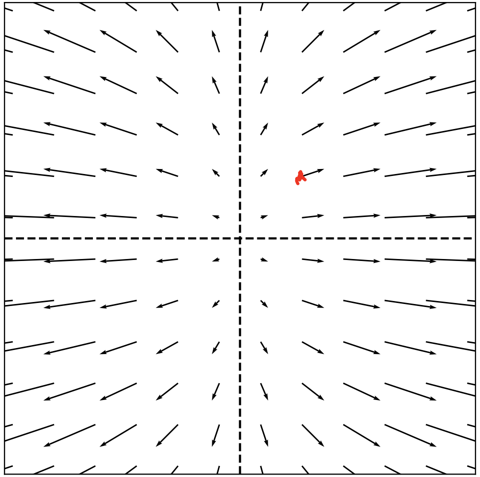
$\vec{x} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$\vec{f}(\vec{x}) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$



Example

- $\vec{f}(\vec{x}) = (3x_1, x_2)^T$

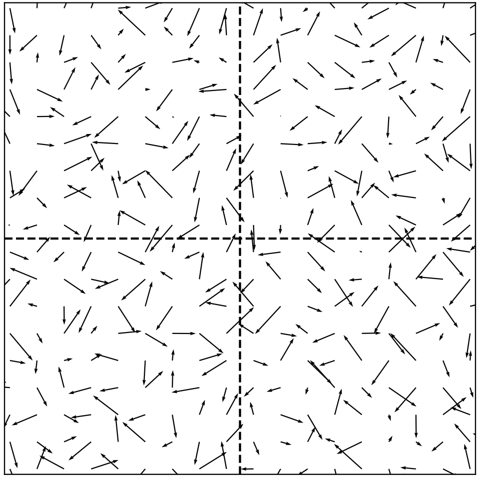


$\vec{x} = (1, 1)$

$\vec{f}(\vec{x}) = ($

Arbitrary Transformations

- Arbitrary transformations can be quite complex.



Arbitrary Transformations

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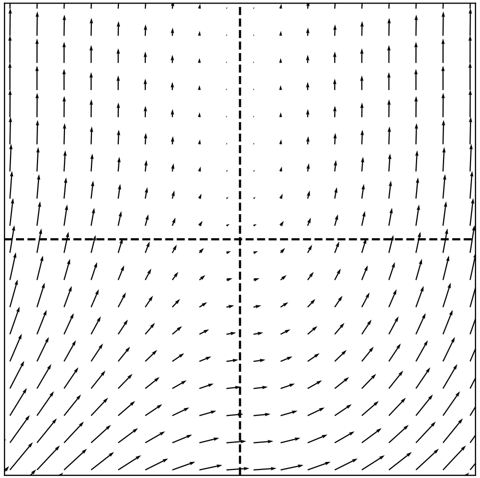


Linear Transformations

- Luckily, we often[1] work with simpler, **linear transformations**.

- A transformation f is linear if:

$$\vec{f}(\alpha\vec{x} + \beta\vec{y}) = \alpha\vec{f}(\vec{x}) + \beta\vec{f}(\vec{y})$$

[1]Sometimes, just to make the math tractable!

Checking Linearity

- To check if a transformation is linear, use the definition.

$$\alpha\vec{x}+\beta\vec{y} = \begin{pmatrix}\alpha x_1+\beta y_1\\ \alpha x_2+\beta y_2\end{pmatrix}$$

- **Example:** $\vec{f}(\vec{x}) = (x_2, -x_1)^T$

$$\vec{f}(\alpha\vec{x}+\beta\vec{y}) = \begin{pmatrix}\alpha x_2+\beta y_2\\ -\alpha x_1-\beta y_1\end{pmatrix}$$

$$\alpha\vec{f}(\vec{x})+\beta\vec{f}(\vec{y}) = \alpha\begin{pmatrix}x_2\\ -x_1\end{pmatrix}+\beta\begin{pmatrix}y_2\\ -y_1\end{pmatrix} = \begin{pmatrix}\alpha x_2+\beta y_2\\ -\alpha x_1-\beta y_1\end{pmatrix}$$

$$x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

f

Exercise

Let $\vec{f}(\vec{x}) = (x_1 + 3, x_2)$. Is $\vec{f}$ a ~~linear~~ transformation?

$$f(\alpha x + \beta y) \overset{?}{=} \alpha f(x) + \beta f(y)$$

$$f(\ldots) = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \alpha \cdot \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3\alpha + 3\beta \\ 0 \end{pmatrix}$$

Implications of Linearity

$\left\{ \vec f(\hat e^{(1)}), \vec f(\hat e^{(2)}) \right.$

- Suppose $\vec f$ is a linear transformation. Then:

$$\begin{aligned}\vec f(\vec x) &= \vec f(x_1 \hat e^{(1)} + x_2 \hat e^{(2)}) \\ &= x_1 \vec f(\hat e^{(1)}) + x_2 \vec f(\hat e^{(2)})\end{aligned}$$

- I.e., $\vec f$ is **totally determined** by what it does to the basis vectors.

The **Complexity** of Arbitrary Transformations

- Suppose f is an **arbitrary** transformation.
- I tell you $\vec{f}(\hat{e}^{(1)}) = (2, 1)^T$ and $\vec{f}(\hat{e}^{(2)}) = (-3, 0)^T$.
- I tell you $\vec{x} = (x_1, x_2)^T$.
- What is $\vec{f}(\vec{x})$?

The Simplicity of Linear Transformations

- Suppose f is a **linear** transformation.
- I tell you $\vec{f}(\hat{e}^{(1)}) = (2, 1)^T$ and $\vec{f}(\hat{e}^{(2)}) = (-3, 0)^T$.
- I tell you $\vec{x} = (x_1, x_2)^T$.
- What is $\vec{f}(\vec{x})$? $= x_1 \vec{f}(\hat{e}^{(1)}) + x_2 \vec{f}(\hat{e}^{(2)})$

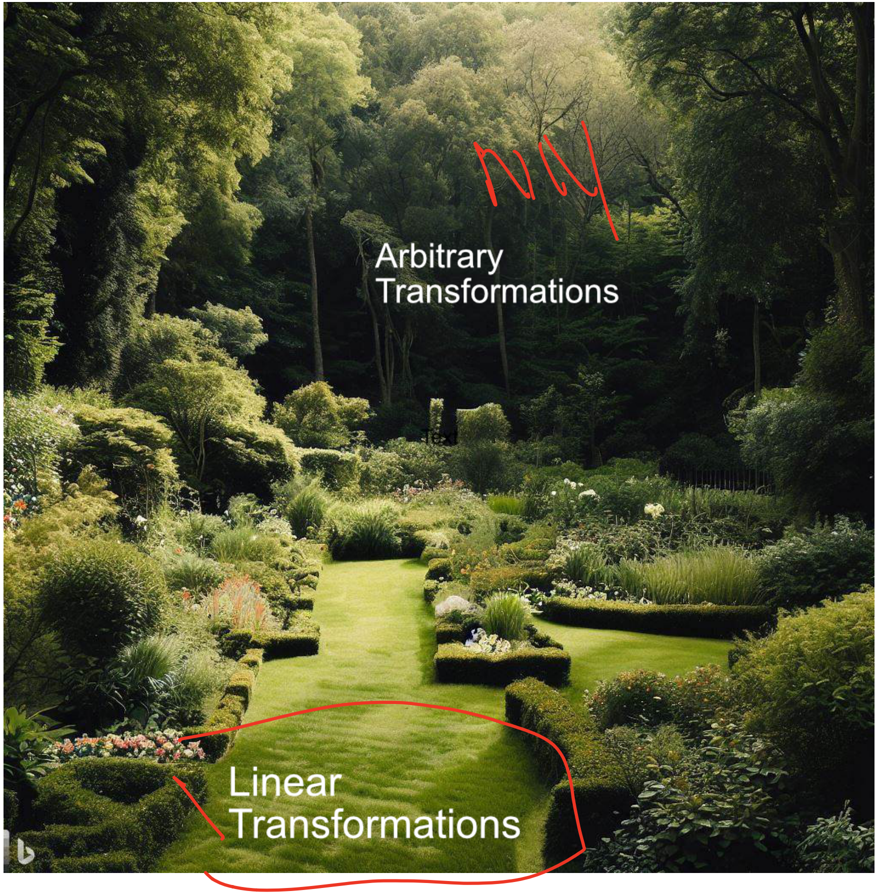
Exercise

- Suppose f is a **linear** transformation.
- I tell you $\vec{f}(\hat{e}^{(1)}) = (2, 1)^T$ and $\vec{f}(\hat{e}^{(2)}) = (-3, 0)^T$.
- I tell you $\vec{x} = (3, -4)^T$.
- What is $\vec{f}(\vec{x})$? $= 3 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} - 4 \begin{pmatrix} -3 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} 6 + 12 \\ 3 - 0 \end{pmatrix} = \begin{pmatrix} 18 \\ 3 \end{pmatrix}$$

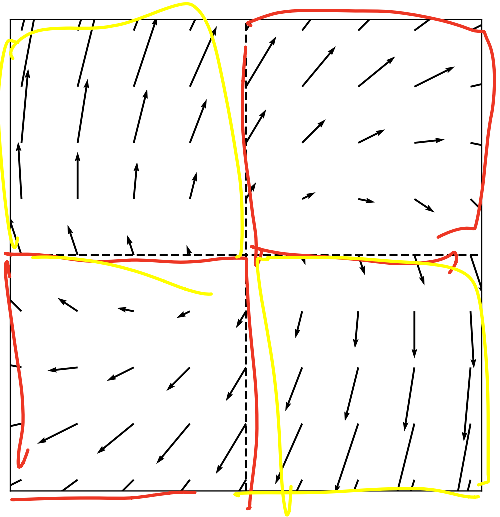
Key Fact

- Linear functions are determined **entirely** by what they do on the basis vectors.
- I.e., to tell you what f does, I only need to tell you $\vec{f}(\hat{e}^{(1)})$ and $\vec{f}(\hat{e}^{(2)})$.
- This makes the math easy!


Arbitrary
Transformations
Text
Linear
Transformations

Example Linear Transformation

- $\vec{f}(\vec{x}) = (x_1 + 3x_2, -3x_1 + 5x_2)^T$

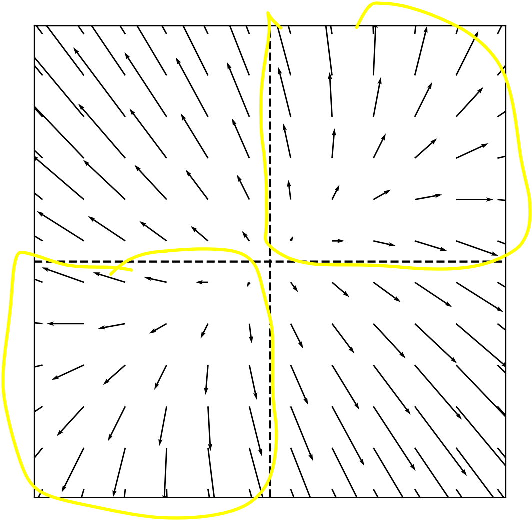


$x = \begin{pmatrix}1\\1\end{pmatrix}$

$f = \begin{pmatrix}4\\2\end{pmatrix}$

Another Example Linear Transformation

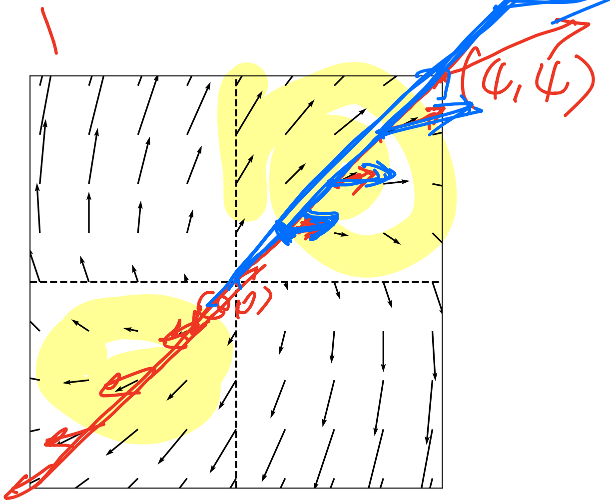
- $\vec{f}(\vec{x}) = (2x_1 - x_2, -x_1 + 3x_2)^T$

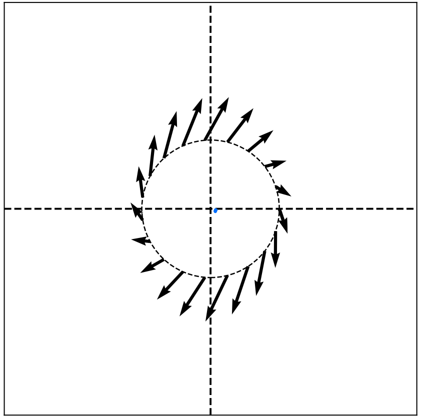


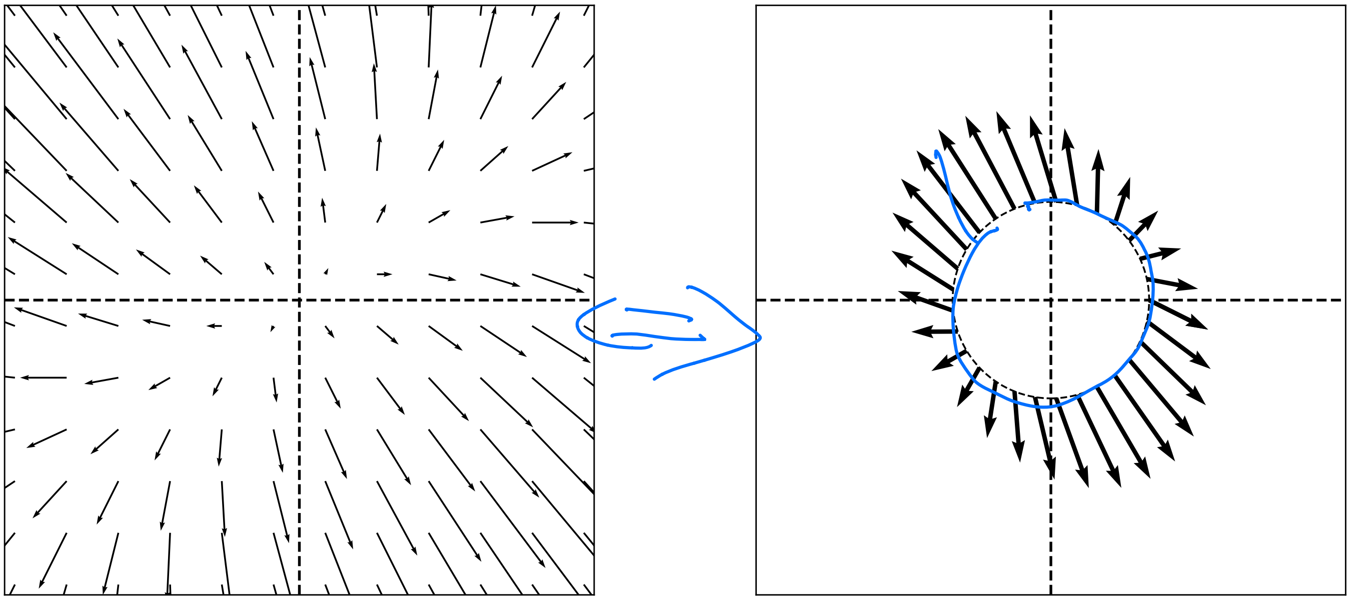
Note

- Because of linearity, along any given direction $\vec{f}$ changes only in scale.

$$\vec{f}(\lambda\hat{x}) = \lambda\vec{f}(\hat{x})$$







Linear Transformations and Bases

- We have been writing transformations in coordinate form. For example:

$$\vec{f}(\vec{x}) = (x_1 + x_2, x_1 - x_2)^T$$

- To do so, we assumed the **standard basis**.

- If we use a different basis, the formula for $\vec{f}$ changes.

Example

- Suppose that in the standard basis, $\vec{f}(\vec{x}) = (x_1 + x_2, x_1 - x_2)^T$.
- Let $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1, 1)^T$ and $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(-1, 1)^T$.
- Write $[\vec{x}]_{\mathcal{U}} = (z_1, z_2)^T$.
- What is $[\vec{f}(\vec{x})]_{\mathcal{U}}$ in terms of z_1 and z_2?

$$\begin{pmatrix} \vec{f}(\vec{x}) \cdot \hat{u}^{(1)} \\ \vec{f}(\vec{x}) \cdot \hat{u}^{(2)} \end{pmatrix}$$

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Lecture 03 | Part 2

Matrices

Matrices?

- I thought this was supposed to be about linear algebra... Where are the matrices?

Matrices?

- I thought this was supposed to be about linear algebra... Where are the matrices?
- What is a matrix, anyways?

What is a matrix?

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Recall: Linear Transformations

- A **transformation** $\vec{f}(\vec{x})$ is a function which takes a vector as input and returns a vector of the same dimensionality.

- A transformation f is **linear** if

$$\vec{f}(\alpha\vec{u} + \beta\vec{v}) = \alpha\vec{f}(\vec{u}) + \beta\vec{f}(\vec{v})$$

Recall: Linear Transformations

- A **key** property: to compute $\vec{f}(\vec{x})$, we only need to know what f does to basis vectors.

- Example:

$$\vec{x} = 3\hat{e}^{(1)} - 4\hat{e}^{(2)} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$\vec{f}(\hat{e}^{(1)}) = -\hat{e}^{(1)} + 3\hat{e}^{(2)}$$

$$\vec{f}(\hat{e}^{(2)}) = 2\hat{e}^{(1)}$$

$$\vec{f}(\vec{x}) =$$

Matrices

- f defined by what it does to basis vectors
- Place $\vec{f}(\hat{e}^{(1)})$, $\vec{f}(\hat{e}^{(2)})$, ... into a table as columns
- This is the **matrix** representing[2] f

$$\vec{f}(\hat{e}^{(1)}) = -\hat{e}^{(1)} + 3\hat{e}^{(2)} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\vec{f}(\hat{e}^{(2)}) = 2\hat{e}^{(1)} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}$$

[2] with respect to the standard basis $\hat{e}^{(1)}, \hat{e}^{(2)}$

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\vec{f}(\hat{e}^{(1)}) = (1, 4, 7)^T$$
$$\vec{f}(\hat{e}^{(2)}) = (2, 5, 7)^T$$
$$\vec{f}(\hat{e}^{(3)}) = (3, 6, 9)^T$$

Main Idea

A square ($n \times n$) matrix can be interpreted as a compact representation of a linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$.

What is matrix multiplication?

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$$

A low-level definition

$$(A\vec{x})_i = \sum_{j=1}^{n} A_{ij} x_j$$

A low-level interpretation

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

In general...

$$\begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \vec{a}^{(1)} & \vec{a}^{(2)} & \vec{a}^{(3)} \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \vec{a}^{(1)} + x_2 \vec{a}^{(2)} + x_3 \vec{a}^{(3)}$$

Matrix Multiplication

$$\vec{x} = x_1\hat{e}^{(1)} + x_2\hat{e}^{(2)} + x_3\hat{e}^{(3)} = (x_1, x_2, x_3)^T$$

$$\vec{f}(\vec{x}) = x_1\vec{f}(\hat{e}^{(1)}) + x_2\vec{f}(\hat{e}^{(2)}) + x_3\vec{f}(\hat{e}^{(3)})$$

$$A = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \vec{f}(\hat{e}^{(1)}) & \vec{f}(\hat{e}^{(2)}) & \vec{f}(\hat{e}^{(3)}) \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

$$A\vec{x} = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \vec{f}(\hat{e}^{(1)}) & \vec{f}(\hat{e}^{(2)}) & \vec{f}(\hat{e}^{(3)}) \\ \downarrow & \downarrow & \downarrow \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= x_1\vec{f}(\hat{e}^{(1)}) + x_2\vec{f}(\hat{e}^{(2)}) + x_3\vec{f}(\hat{e}^{(3)})$$

Matrix Multiplication

- Matrix A represents a linear transformation $\vec{f}$
  - With respect to the standard basis
  - If we use a different basis, the matrix changes!

- Matrix multiplication $A\vec{x}$ **evaluates** $\vec{f}(\vec{x})$

What are they, *really*?

- Matrices are sometimes just tables of numbers.
- But they often have a deeper meaning.

Main Idea

A square ($n \times n$) matrix can be interpreted as a compact representation of a linear transformation $\vec{f} : \mathbb{R}^n \to \mathbb{R}^n$.

What's more, if A represents $\vec{f}$, then $A\vec{x} = \vec{f}(\vec{x})$; that is, multiplying by A is the same as evaluating $\vec{f}$.

Example

$$\vec{x} = 3\hat{e}^{(1)} - 4\hat{e}^{(2)} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$\vec{f}(\hat{e}^{(1)}) = -\hat{e}^{(1)} + 3\hat{e}^{(2)}$$

$$\vec{f}(\hat{e}^{(2)}) = 2\hat{e}^{(1)}$$

$$\vec{f}(\vec{x}) =$$

$$A =$$

$$A\vec{x} =$$

Note

- All of this works because we assumed $\vec{f}$ is **linear**.
- If it isn't, evaluating $\vec{f}$ isn't so simple.

Note

- All of this works because we assumed $\vec{f}$ is **linear**.
- If it isn't, evaluating $\vec{f}$ isn't so simple.
- Linear algebra = simple!