

DSC 140B

Representation Learning

Lecture 04 | Part 1

Matrices

Matrices?

- ▶ I thought this week was supposed to be about linear algebra... Where are the matrices?

Matrices?


- ▶ I thought this week was supposed to be about linear algebra... Where are the matrices?
- ▶ What is a matrix, anyways?

What is a matrix?

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Recall: Linear Transformations

- ▶ A **transformation** $\vec{f}(\vec{x})$ is a function which takes a vector as input and returns a vector of the same dimensionality.
- ▶ A transformation \vec{f} is **linear** if

$$\vec{f}(\alpha\vec{u} + \beta\vec{v}) = \alpha\vec{f}(\vec{u}) + \beta\vec{f}(\vec{v})$$


Recall: Linear Transformations

$$\begin{pmatrix} -11 \\ 9 \end{pmatrix}$$

- ▶ **Key** consequence of **linearity**: to compute $\vec{f}(\vec{x})$, only need to know what \vec{f} does to basis vectors.

$$\vec{f}(\vec{x}) = \vec{f}(\alpha \hat{e}^{(1)} + \beta \hat{e}^{(2)}) = \alpha \vec{f}(\hat{e}^{(1)}) + \beta \vec{f}(\hat{e}^{(2)})$$

- ▶ Example:

$$\vec{x} = 3\hat{e}^{(1)} - 4\hat{e}^{(2)} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$\vec{f}(\hat{e}^{(1)}) = -\hat{e}^{(1)} + 3\hat{e}^{(2)}$$

$$\vec{f}(\hat{e}^{(2)}) = 2\hat{e}^{(1)}$$

$$\begin{aligned} \vec{f}(\vec{x}) &= \vec{f}(3\hat{e}^{(1)} - 4\hat{e}^{(2)}) = 3\vec{f}(\hat{e}^{(1)}) - 4\vec{f}(\hat{e}^{(2)}) \\ &= 3(-\hat{e}^{(1)} + 3\hat{e}^{(2)}) - 4(2\hat{e}^{(1)}) = -3\hat{e}^{(1)} + 9\hat{e}^{(2)} - 8\hat{e}^{(1)} = -11\hat{e}^{(1)} + 9\hat{e}^{(2)} \end{aligned}$$

$$2\alpha + \beta \vec{f}(\hat{e}^{(2)})$$

Matrices

- ▶ **Idea:** Since \vec{f} is defined by what it does to basis, place $\vec{f}(\hat{e}^{(1)})$, $\vec{f}(\hat{e}^{(2)})$, ... into a table as columns
- ▶ This is the **matrix** representing¹ \vec{f}

$$\left\{ \begin{array}{l} \vec{f}(\hat{e}^{(1)}) = -\hat{e}^{(1)} + 3\hat{e}^{(2)} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \\ \vec{f}(\hat{e}^{(2)}) = 2\hat{e}^{(1)} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \end{array} \right.$$

$$\begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}$$

¹with respect to the standard basis $\hat{e}^{(1)}, \hat{e}^{(2)}$

Exercise

Write the matrix representing \vec{f} with respect to the standard basis, given:

$$\vec{f}(\hat{e}^{(1)}) = (1, 4, 7)^T$$

$$\vec{f}(\hat{e}^{(2)}) = (2, 5, 7)^T$$

$$\vec{f}(\hat{e}^{(3)}) = (3, 6, 9)^T$$

A handwritten matrix in red ink is shown to the right of the text. The matrix is a 3x3 matrix with columns corresponding to the standard basis vectors $\hat{e}^{(1)}$, $\hat{e}^{(2)}$, and $\hat{e}^{(3)}$. The entries are: first column (1, 4, 7), second column (2, 5, 7), and third column (3, 6, 9). Arrows point from the labels $\vec{f}(\hat{e}^{(1)})$, $\vec{f}(\hat{e}^{(2)})$, and $\vec{f}(\hat{e}^{(3)})$ to the respective columns of the matrix.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 7 & 9 \end{pmatrix}$$

Exercise

Suppose \vec{f} has the matrix below:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = (\vec{f}(e^{(1)}), \vec{f}(e^{(2)}), \vec{f}(e^{(3)}))$$

Let $\vec{x} = (-2, 1, 3)^T$. What is $\vec{f}(\vec{x})$?

$$\vec{f}(\vec{x}) = -2\vec{f}(e^{(1)}) + \vec{f}(e^{(2)}) + 3\vec{f}(e^{(3)})$$

Main Idea

A square $(n \times n)$ matrix can be interpreted as a compact representation of a linear transformation $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

What is matrix multiplication?

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \text{circled} \\ \text{circled} \\ \text{circled} \end{pmatrix}$$

$x_i^T y = 1x - 2 + 2x + 3x$

$3 \times 3 \quad 3 \times 1 \quad 3 \times 1$

$n \times m \quad m \times q \quad n \times q$

A low-level definition

$$A \cdot \vec{x} = \vec{y}$$

$n \times n$ $n \times 1$ $n \times 1$

$$(A\vec{x})_i = \sum_{j=1}^n A_{ij} x_j$$

$i = 1, \dots, n$

$$\sum_{j=1}^n A_{ij} \cdot x_j$$

A low-level interpretation

The diagram illustrates the low-level interpretation of matrix-vector multiplication. On the left, a 3x3 matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ and a column vector $\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ are shown. Red and purple lines connect the elements of the matrix to the elements of the vector, showing how each row of the matrix is multiplied by the vector. On the right, the result is expressed as a sum of three column vectors: $-2 \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$. Handwritten purple annotations show the calculation of the first row: $-2 \cdot 1 + 1 \cdot 2 + 3 \cdot 3$ and the calculation of the second row: $-2 \cdot 4 + 1 \cdot 5 + 3 \cdot 6$.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

$-2 \cdot 1 + 1 \cdot 2 + 3 \cdot 3$

$-2 \cdot 4 + 1 \cdot 5 + 3 \cdot 6$

In general...

$$\left(\begin{array}{c} \uparrow \\ \vec{a}^{(1)} \\ \downarrow \end{array} \right) \left(\begin{array}{c} \uparrow \\ \vec{a}^{(2)} \\ \downarrow \end{array} \right) \left(\begin{array}{c} \uparrow \\ \vec{a}^{(3)} \\ \downarrow \end{array} \right) \left(\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right) = x_1 \vec{a}^{(1)} + x_2 \vec{a}^{(2)} + x_3 \vec{a}^{(3)}$$



Matrix Multiplication

$$\vec{f}(\vec{x}) = A \vec{x}$$

$$\vec{x} = x_1 \hat{e}^{(1)} + x_2 \hat{e}^{(2)} + x_3 \hat{e}^{(3)} = (x_1, x_2, x_3)^T$$

$$\vec{f}(\vec{x}) = x_1 \vec{f}(\hat{e}^{(1)}) + x_2 \vec{f}(\hat{e}^{(2)}) + x_3 \vec{f}(\hat{e}^{(3)})$$

$n \times n$

$$A = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \vec{f}(\hat{e}^{(1)}) & \vec{f}(\hat{e}^{(2)}) & \vec{f}(\hat{e}^{(3)}) \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

$$A\vec{x} = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \vec{f}(\hat{e}^{(1)}) & \vec{f}(\hat{e}^{(2)}) & \vec{f}(\hat{e}^{(3)}) \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= x_1 \vec{f}(\hat{e}^{(1)}) + x_2 \vec{f}(\hat{e}^{(2)}) + x_3 \vec{f}(\hat{e}^{(3)})$$

Matrix Multiplication

- ▶ Matrix A represents a linear transformation \vec{f}
 - ▶ With respect to the standard basis
 - ▶ If we use a different basis, the matrix changes!
- ▶ Matrix multiplication $A\vec{x}$ **evaluates** $\vec{f}(\vec{x})$

$$A\vec{x} = \vec{f}(\vec{x})$$

What are they, *really*?

- ▶ Matrices are sometimes just tables of numbers.
- ▶ But they often have a deeper meaning.

Main Idea

A square ($n \times n$) matrix can be interpreted as a compact representation of a linear transformation $\vec{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

What's more, if A represents \vec{f} , then $A\vec{x} = \vec{f}(\vec{x})$; that is, multiplying by A is the same as evaluating \vec{f} .

Example

$$\vec{x} = 3\hat{e}^{(1)} - 4\hat{e}^{(2)} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$\vec{f}(\hat{e}^{(1)}) = -\hat{e}^{(1)} + 3\hat{e}^{(2)}$$

$$\vec{f}(\hat{e}^{(2)}) = 2\hat{e}^{(1)}$$

$$\vec{f}(\vec{x}) =$$



$$A = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}$$

$$A\vec{x} = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} -11 \\ 9 \end{pmatrix}$$

Note

- ▶ All of this works because we assumed \vec{f} is **linear**.
- ▶ If it isn't, evaluating \vec{f} isn't so simple.

Note

- ▶ All of this works because we assumed \vec{f} is **linear**.
- ▶ If it isn't, evaluating \vec{f} isn't so simple.
- ▶ Linear algebra = simple!

Matrices in Other Bases

- ▶ The matrix of a linear transformation wrt the standard basis:

$$\begin{pmatrix} \uparrow & \uparrow & \uparrow & \\ \vec{f}(\hat{e}^{(1)}) & \vec{f}(\hat{e}^{(2)}) & \dots & \vec{f}(\hat{e}^{(d)}) \\ \downarrow & \downarrow & \downarrow & \end{pmatrix}$$

(Note: Red wavy lines are drawn under each column vector in the matrix.)

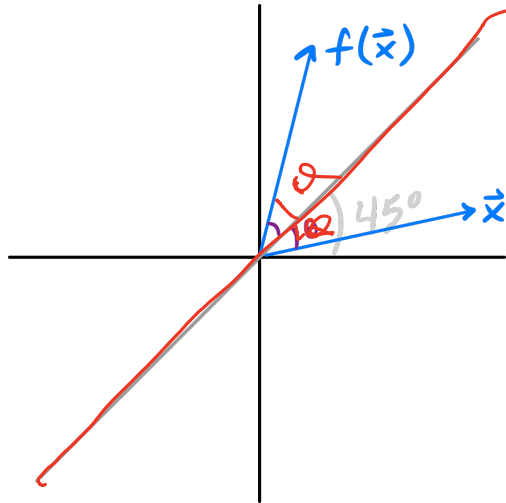
- ▶ With respect to basis \mathcal{U} :

$$\begin{pmatrix} \uparrow & \uparrow & \uparrow & \\ [\vec{f}(\hat{u}^{(1)})]_{\mathcal{U}} & [\vec{f}(\hat{u}^{(2)})]_{\mathcal{U}} & \dots & [\vec{f}(\hat{u}^{(d)})]_{\mathcal{U}} \\ \downarrow & \downarrow & \downarrow & \end{pmatrix}$$

(Note: Red wavy lines are drawn under each column vector in the matrix. A red arrow points from the text 'basis U' to the first column.)

Matrices in Other Bases

- ▶ Consider the transformation \vec{f} which “mirrors” a vector over the line of 45° .



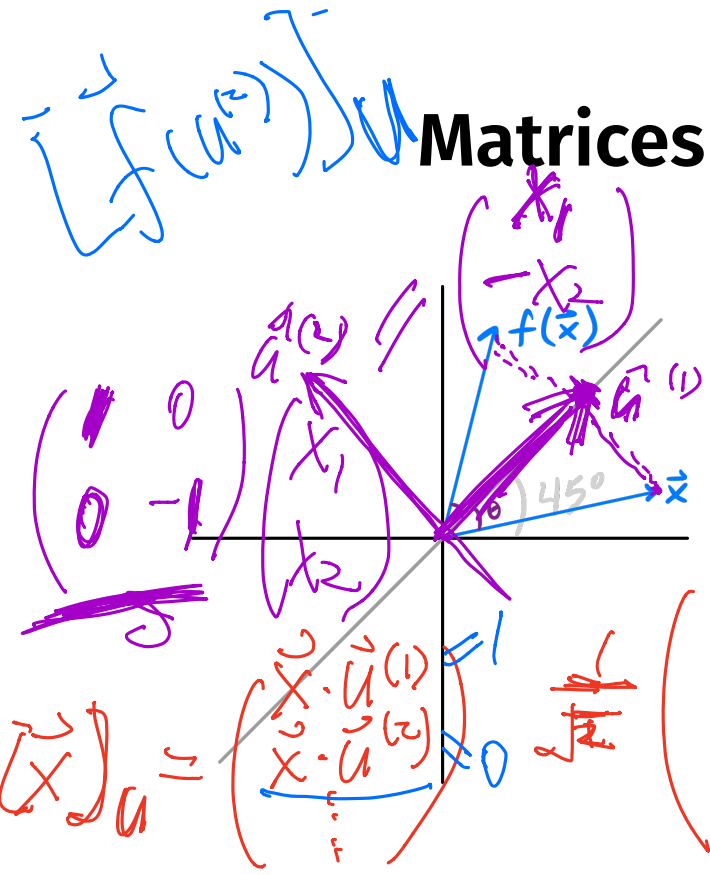
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$f(\vec{e}_i)$$

- ▶ What is its matrix in the standard basis?

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Matrices in Other Bases



$$f(\hat{u}^{(1)}) = A \cdot \hat{u}^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- ▶ Let $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1, 1)^T$
- ▶ Let $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(-1, 1)^T$
- ▶ What is $[f(\hat{u}^{(1)})]_{\mathcal{U}}$?
- ▶ $[f(\hat{u}^{(2)})]_{\mathcal{U}}$?
- ▶ What is the matrix?

$$A_{\mathcal{U}\mathcal{U}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

DSC 140B

Representation Learning

Lecture 04 | Part 2

The Spectral Theorem

Eigenvectors

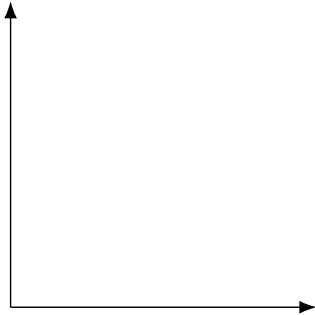
- ▶ Let A be an $n \times n$ matrix. An **eigenvector** of A with **eigenvalue** λ is a nonzero vector \vec{v} such that $A\vec{v} = \lambda\vec{v}$.

Eigenvectors (of Linear Transformations)

- ▶ Let \vec{f} be a linear transformation. An **eigenvector** of \vec{f} with **eigenvalue** λ is a nonzero vector \vec{v} such that $f(\vec{v}) = \lambda\vec{v}$.

Geometric Interpretation

- ▶ When \vec{f} is applied to one of its eigenvectors, \vec{f} simply scales it.
 - ▶ Possibly by a negative amount.



Symmetric Matrices

- ▶ Recall: a matrix A is **symmetric** if $A^T = A$.

The Spectral Theorem²

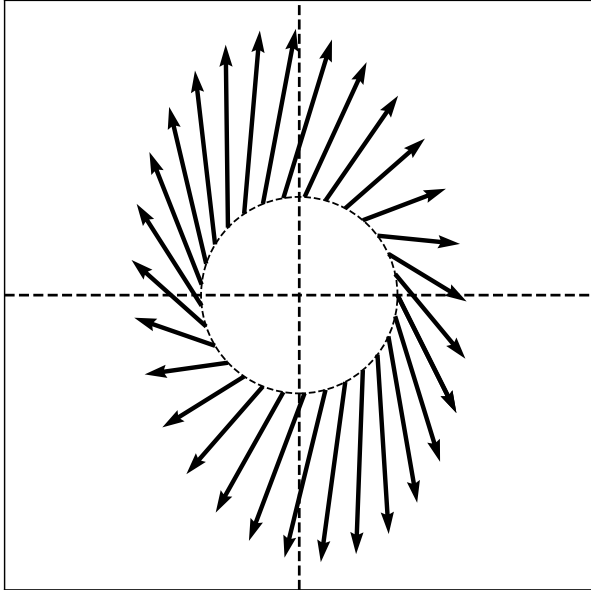
- ▶ **Theorem:** Let A be an $n \times n$ *symmetric* matrix. Then there exist n eigenvectors of A which are all mutually orthogonal.

²for symmetric matrices

What?

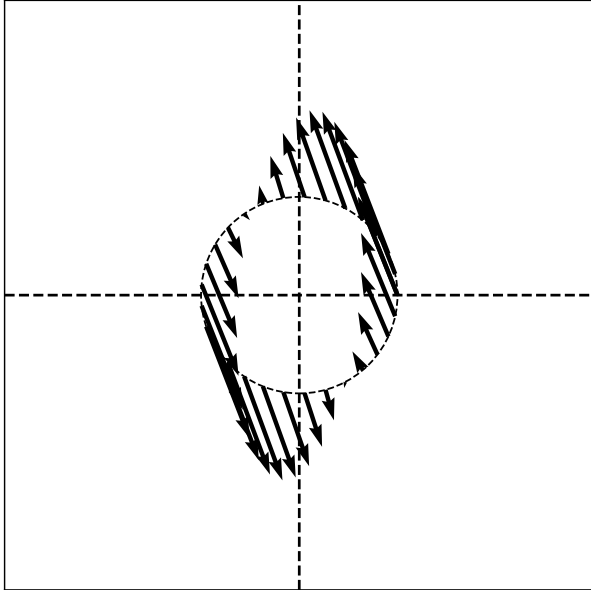
- ▶ What does the spectral theorem mean?
- ▶ What is an eigenvector, really?
- ▶ Why are they useful?

Example Linear Transformation



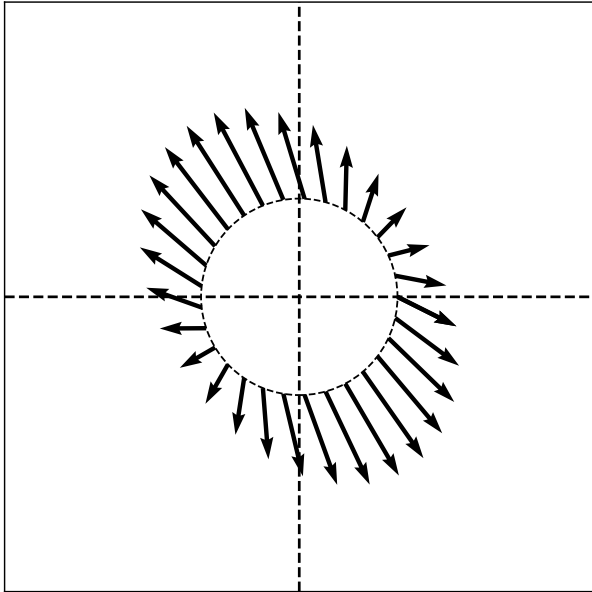
$$A = \begin{pmatrix} 5 & 5 \\ -10 & 12 \end{pmatrix}$$

Example Linear Transformation



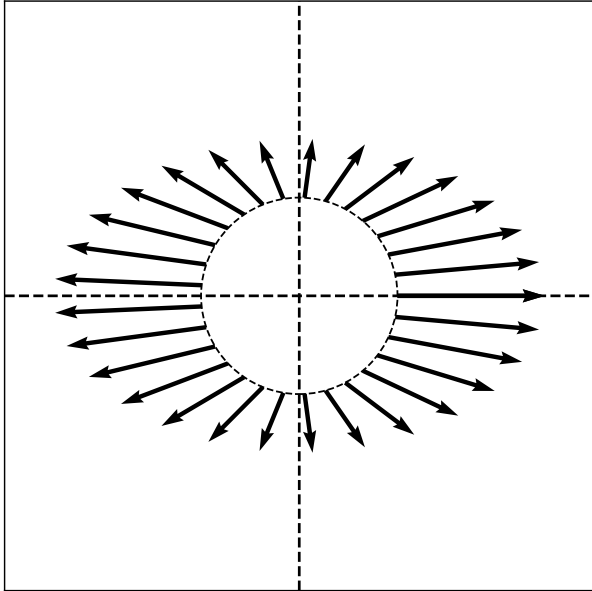
$$A = \begin{pmatrix} -2 & -1 \\ -5 & 3 \end{pmatrix}$$

Example Symmetric Linear Transformation



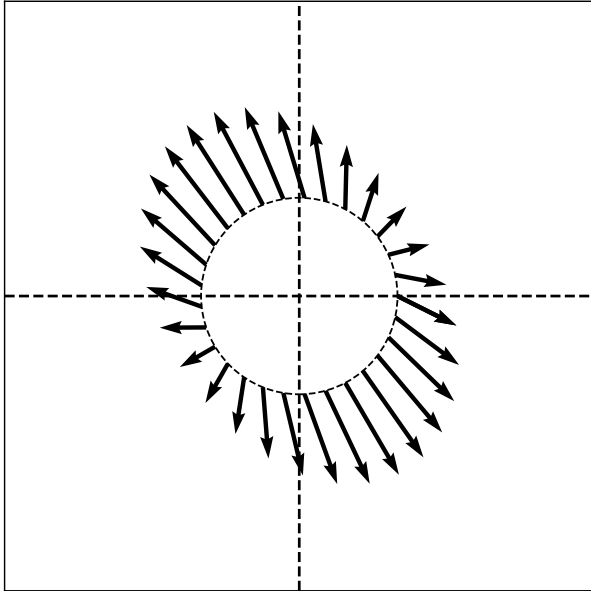
$$A = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

Example Symmetric Linear Transformation



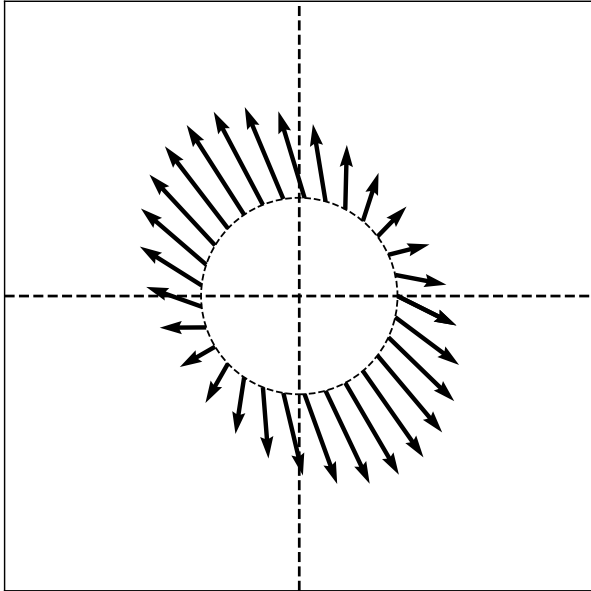
$$A = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$$

Observation #1



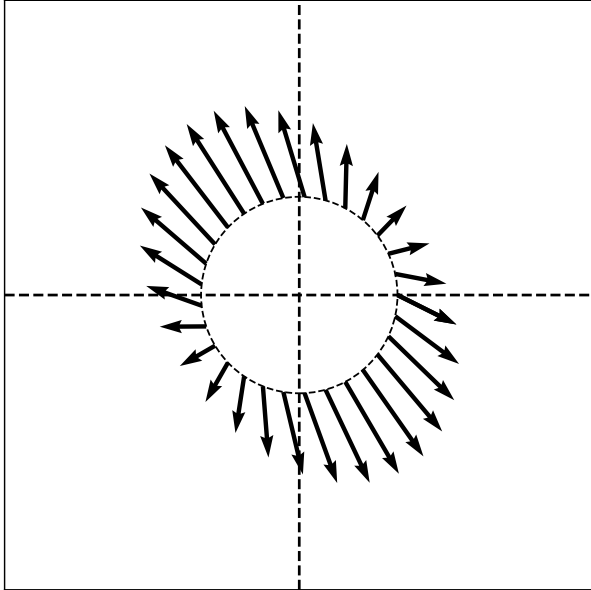
- ▶ Symmetric linear transformations have **axes of symmetry.**

Observation #2



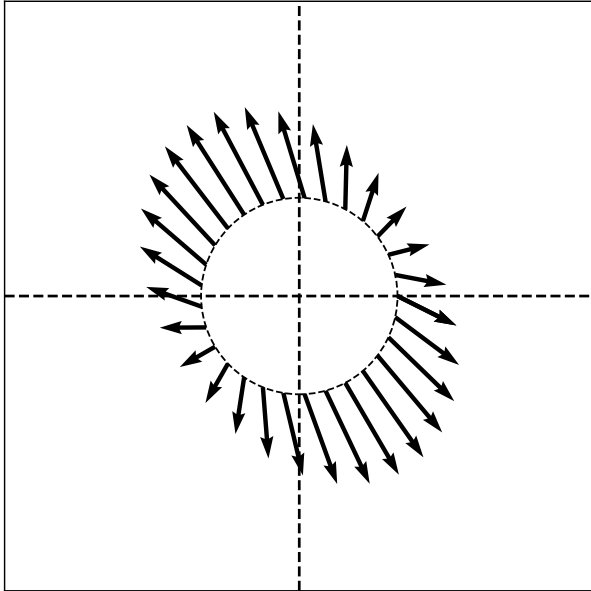
- ▶ The axes of symmetry are **orthogonal** to one another.

Observation #3



- ▶ The action of \vec{f} along an axis of symmetry is simply to scale its input.

Observation #4



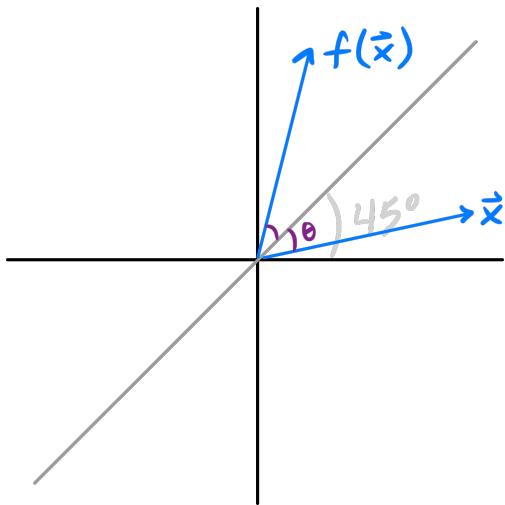
- ▶ The size of this scaling can be different for each axis.

Main Idea

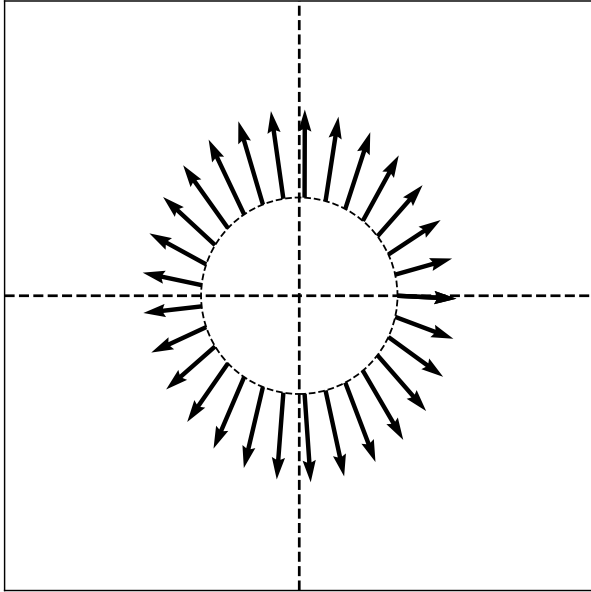
The **eigenvectors** of a symmetric linear transformation (matrix) are its axes of symmetry. The **eigenvalues** describe how much each axis of symmetry is scaled.

Exercise

Consider the linear transformation which mirrors its input over the line of 45° . Give two orthogonal eigenvector of the transformation.

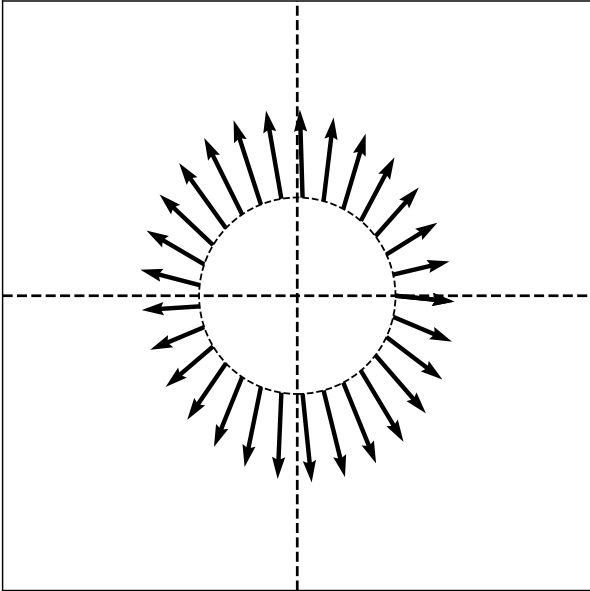


Off-diagonal elements



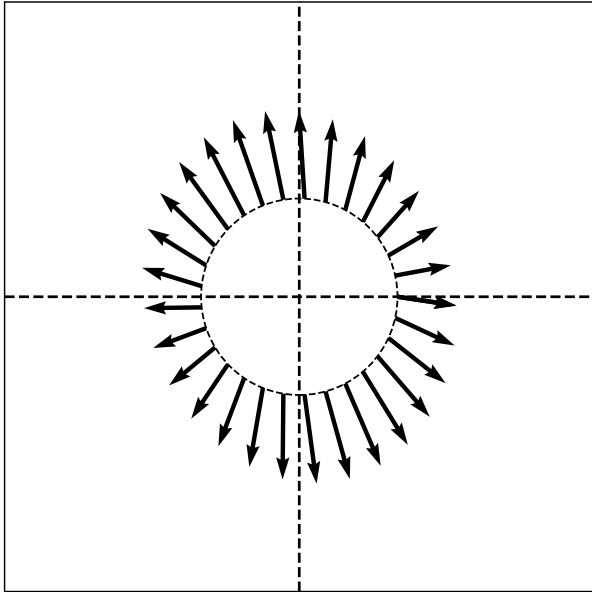
$$A = \begin{pmatrix} 5 & -0.1 \\ -0.1 & 2 \end{pmatrix}$$

Off-diagonal elements



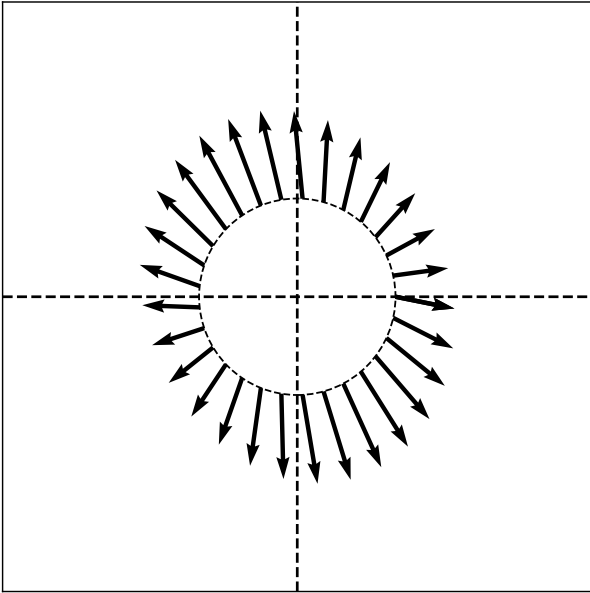
$$A = \begin{pmatrix} 5 & -0.2 \\ -0.2 & 2 \end{pmatrix}$$

Off-diagonal elements



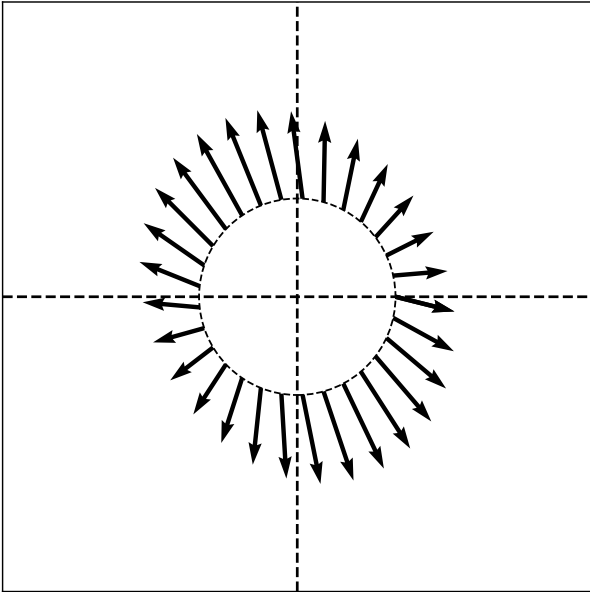
$$A = \begin{pmatrix} 5 & -0.3 \\ -0.3 & 2 \end{pmatrix}$$

Off-diagonal elements



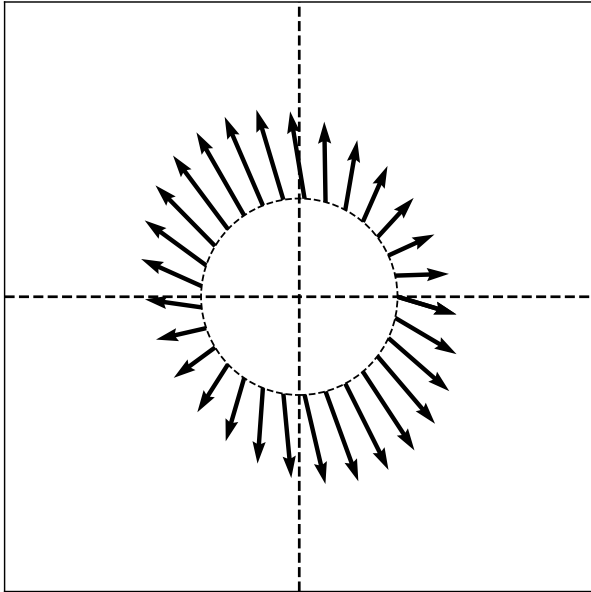
$$A = \begin{pmatrix} 5 & -0.4 \\ -0.4 & 2 \end{pmatrix}$$

Off-diagonal elements



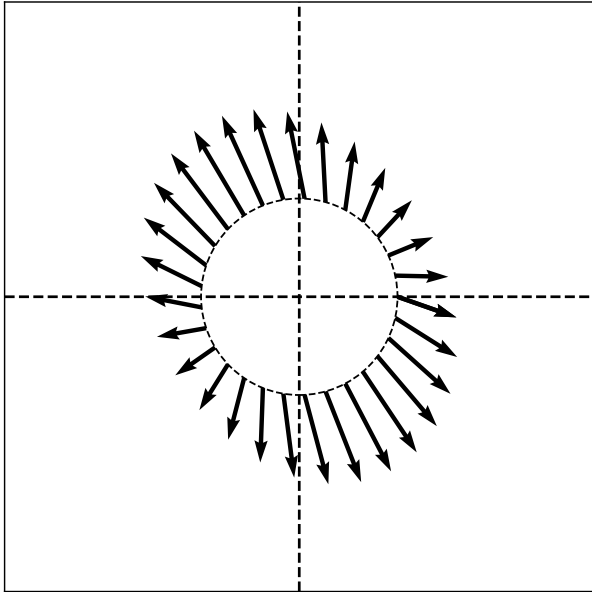
$$A = \begin{pmatrix} 5 & -0.5 \\ -0.5 & 2 \end{pmatrix}$$

Off-diagonal elements



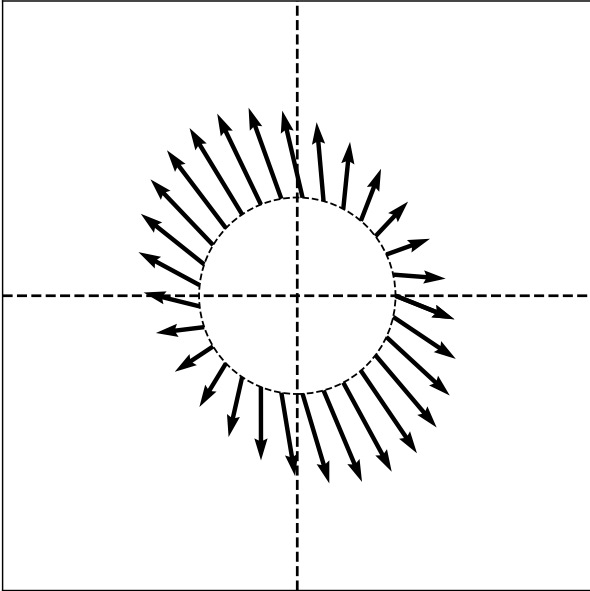
$$A = \begin{pmatrix} 5 & -0.6 \\ -0.6 & 2 \end{pmatrix}$$

Off-diagonal elements



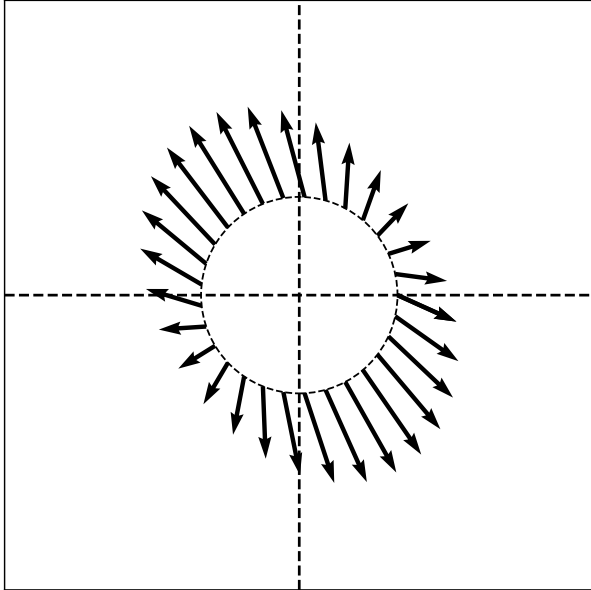
$$A = \begin{pmatrix} 5 & -0.7 \\ -0.7 & 2 \end{pmatrix}$$

Off-diagonal elements



$$A = \begin{pmatrix} 5 & -0.8 \\ -0.8 & 2 \end{pmatrix}$$

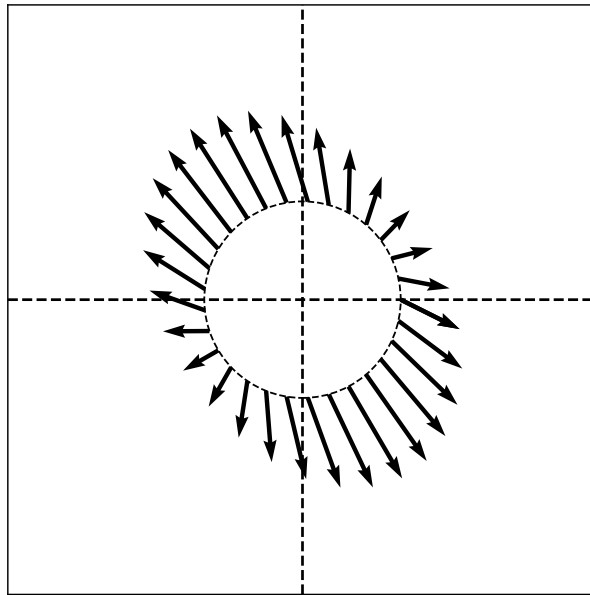
Off-diagonal elements



$$A = \begin{pmatrix} 5 & -0.9 \\ -0.9 & 2 \end{pmatrix}$$

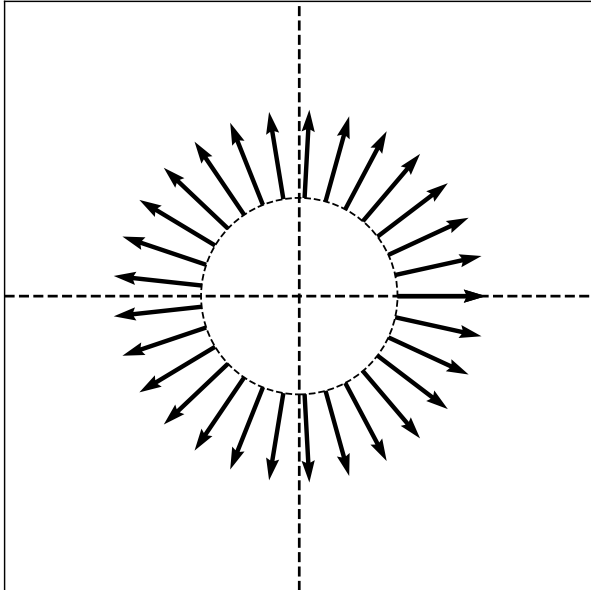
The Spectral Theorem³

- ▶ **Theorem:** Let A be an $n \times n$ symmetric matrix. Then there exist n eigenvectors of A which are all mutually orthogonal.



³for symmetric matrices

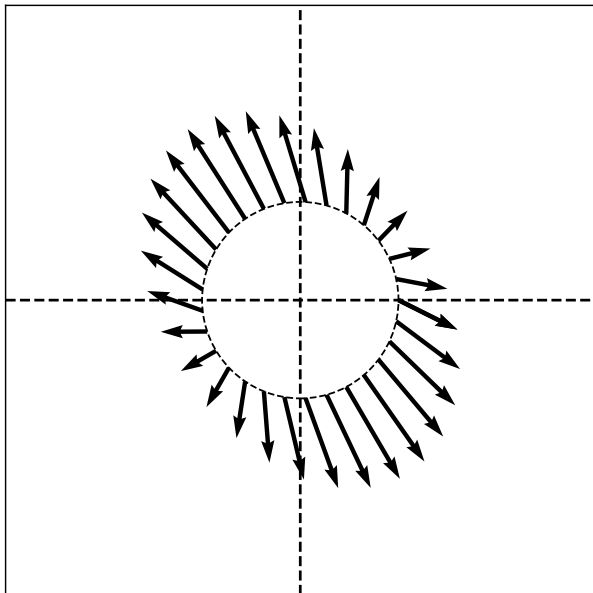
What about total symmetry?



- ▶ Every vector is an eigenvector.

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

Computing Eigenvectors



```
>> A = np.array([[2, -1], [-1, 3]])  
>> np.linalg.eigh(A)  
(array([1.38196601, 3.61803399]),  
 array([[ -0.85065081, -0.52573111],  
        [ -0.52573111,  0.85065081]]))
```