DSC 140B

Representation Learning

Lecture 03 | Part 1

Functions of a Vector

Functions of a Vector

- In ML, we often work with functions of a vector: $f : \mathbb{R}^d \to \mathbb{R}^{d'}$.

- Example: a prediction function, $H(\vec{x})$.

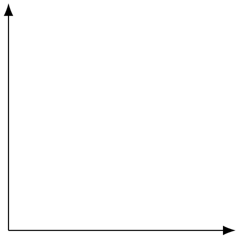
- Functions of a vector can return:
  - a number: $f : \mathbb{R}^d \to \mathbb{R}^1$
  - a vector $\vec{f} : \mathbb{R}^d \to \mathbb{R}^{d'}$
  - something else?

Transformations

- A **transformation** $\vec{f}$ is a function that takes in a vector, and returns a vector *of the same dimensionality*.
- That is, $\vec{f} : \mathbb{R}^d \to \mathbb{R}^d$.

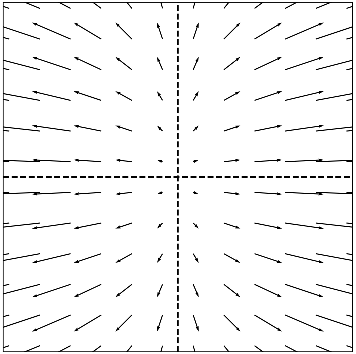
Visualizing Transformations

- A transformation is a **vector field**.
  - Assigns a vector to each point in space.
  - Example: $\vec{f}(\vec{x}) = (3x_1, x_2)^T$



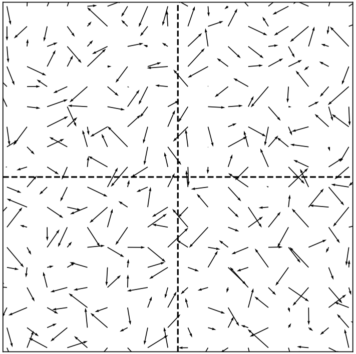
Example

- $\vec{f}(\vec{x}) = (3x_1, x_2)^T$



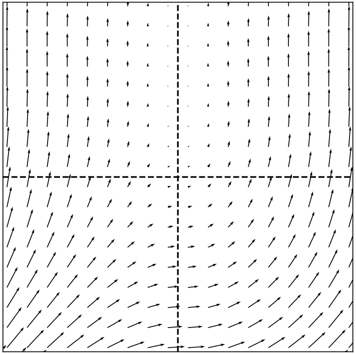
Arbitrary Transformations

- Arbitrary transformations can be quite complex.



Arbitrary Transformations

- Arbitrary transformations can be quite complex.



Linear Transformations

Linear

- Luckily, we often[1] work with simpler, **linear transformations**.
- A transformation f is linear if:

$$\vec{f}(\alpha\vec{x} + \beta\vec{y}) = \alpha\vec{f}(\vec{x}) + \beta\vec{f}(\vec{y})$$

[1] Sometimes, just to make the math tractable!

Checking Linearity

- To check if a transformation is linear, use the definition.

- **Example:** $\vec{f}(\vec{x}) = (x_2, -x_1)^T$

$$\vec{f}(\alpha\vec{x} + \beta\vec{y}) = \cdots\cdots = \alpha f(\vec{x}) + \beta(\vec{y})$$

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \quad \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix}$$

$\alpha \ \vec{a} \quad \beta \vec{b}$

$$\vec{f}(\alpha\vec{a}+\beta\vec{b}) \neq \alpha f(\vec{a})+\beta f(\vec{b})$$

$\alpha = 2$ $\quad \vec{a} = \begin{pmatrix}1\\1\end{pmatrix} \quad \vec{b} = \begin{pmatrix}2\\3\end{pmatrix}$

$\beta = 3$

Exercise

Let $\vec{f}(\vec{x}) = (x_1 + 3, x_2)$. Is $\vec{f}$ a linear transformation?

$$\alpha\vec{a}+\beta\vec{b} = 2\begin{pmatrix}1\\1\end{pmatrix} + 3\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}+\begin{pmatrix}6\\9\end{pmatrix} = \begin{pmatrix}8\\11\end{pmatrix}$$

$$\vec{f}(\alpha\vec{a}+\beta\vec{b}) = \vec{f}\left(\begin{pmatrix}8\\11\end{pmatrix}\right) = \begin{pmatrix}11\\11\end{pmatrix}$$

$$\alpha f(\vec{a})+\beta f(\vec{b}) = 2f\begin{pmatrix}1\\1\end{pmatrix} + 3f\begin{pmatrix}2\\3\end{pmatrix} = 2\begin{pmatrix}4\\1\end{pmatrix}+3\begin{pmatrix}5\\3\end{pmatrix} = \begin{pmatrix}23\\11\end{pmatrix}$$

$\alpha = 2 \quad \vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\beta = 3$

Exercise

Let $\vec{f}(\vec{x}) = (x_1 + 3, x_2)$. Is $\vec{f}$ a linear transformation?

$\alpha\vec{a} + \beta\vec{b} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$

$\vec{f}(\alpha\vec{a} + \beta\vec{b}) = \vec{f}\left(\begin{pmatrix} 8 \\ 5 \end{pmatrix}\right) = \begin{pmatrix} 11 \\ 5 \end{pmatrix}$

$\alpha f(\vec{a}) + \beta f(\vec{b}) \quad 2f\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3f\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 23 \\ 5 \end{pmatrix}$

Implications of Linearity

- Suppose $\vec{f}$ is a linear transformation. Then:

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\begin{aligned}\vec{f}(\vec{x}) &= \vec{f}(x_1 \hat{e}^{(1)} + x_2 \hat{e}^{(2)}) \\ &= x_1 \vec{f}(\hat{e}^{(1)}) + x_2 \vec{f}(\hat{e}^{(2)})\end{aligned}$$

- I.e., $\vec{f}$ is **totally determined** by what it does to the basis vectors.

The **Complexity** of Arbitrary Transformations

- Suppose f is an **arbitrary** transformation.
- I tell you $\vec{f}(\hat{e}^{(1)}) = (2, 1)^T$ and $\vec{f}(\hat{e}^{(2)}) = (-3, 0)^T$.
- I tell you $\vec{x} = (x_1, x_2)^T$.
- What is $\vec{f}(\vec{x})$?

The **Simplicity** of Linear Transformations

- Suppose f is a **linear** transformation.
- I tell you $\vec{f}(\hat{e}^{(1)}) = (2, 1)^T$ and $\vec{f}(\hat{e}^{(2)}) = (-3, 0)^T$.
- I tell you $\vec{x} = (x_1, x_2)^T$.
- What is $\vec{f}(\vec{x})$?

$$\vec{f}(\vec{x}) = \vec{f}(x_1 \hat{e}^{(1)} + x_2 \hat{e}^{(2)}) = x_1 f(\hat{e}^{(1)}) + x_2 f(\hat{e}^{(2)})$$

because of linearity!

Exercise

- Suppose f is a **linear** transformation.
- I tell you $\vec{f}(\hat{e}^{(1)}) = (2, 1)^T$ and $\vec{f}(\hat{e}^{(2)}) = (-3, 0)^T$.
- I tell you $\vec{x} = (3, -4)^T$.
- What is $\vec{f}(\vec{x})$?

$$f\left(\begin{pmatrix} 3 \\ -4 \end{pmatrix}\right) = 3f(\hat{e}^{(1)}) - 4f(\hat{e}^{(2)})$$
$$= 3\begin{pmatrix} 2 \\ 1 \end{pmatrix} - 4\begin{pmatrix} -3 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 6 \\ 3 \end{pmatrix} + \begin{pmatrix} 12 \\ 0 \end{pmatrix} = \begin{pmatrix} 18 \\ 3 \end{pmatrix}$$

$$f\begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2f(\hat{e}^{(1)}) - 1f(\hat{e}^{(2)})$$
$$= 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} - 1\begin{pmatrix} -3 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$$

Exercise

- Suppose f is a **linear** transformation.
- I tell you $\vec{f}(\hat{e}^{(1)}) = (2, 1)^T$ and $\vec{f}(\hat{e}^{(2)}) = (-3, 0)^T$.
- I tell you $\vec{x} = (3, -4)^T$.
- What is $\vec{f}(\vec{x})$?

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\vec{f}(\vec{x}) = x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

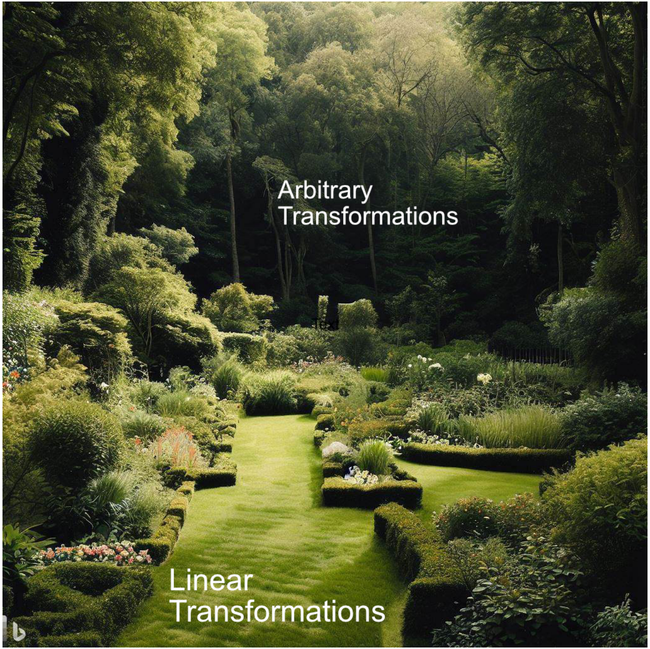
$$= \begin{pmatrix} 2x_1 - 3x_2 \\ x_1 \end{pmatrix}$$

Key Fact

- Linear functions are determined **entirely** by what they do on the basis vectors.

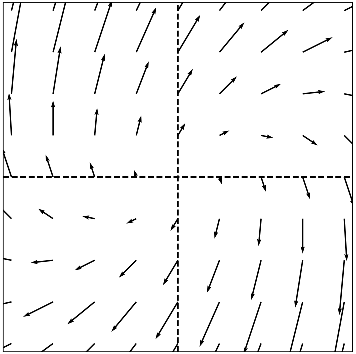
- I.e., to tell you what f does, I only need to tell you $\vec{f}(\hat{e}^{(1)})$ and $\vec{f}(\hat{e}^{(2)})$.

- This makes the math easy!


Arbitrary Transformations
Linear Transformations

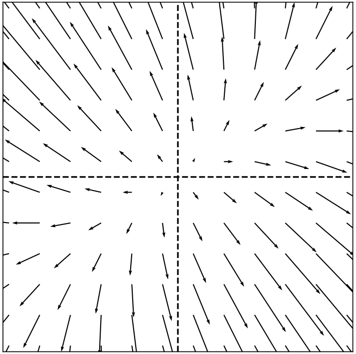
Example Linear Transformation

- $\vec{f}(\vec{x}) = (x_1 + 3x_2, -3x_1 + 5x_2)^T$



Another Example Linear Transformation

- $\vec{f}(\vec{x}) = (2x_1 - x_2, -x_1 + 3x_2)^T$



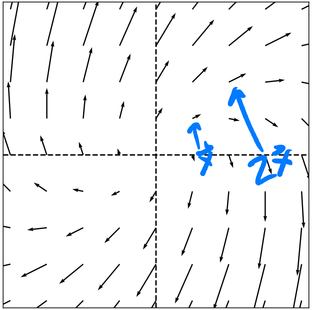
Note

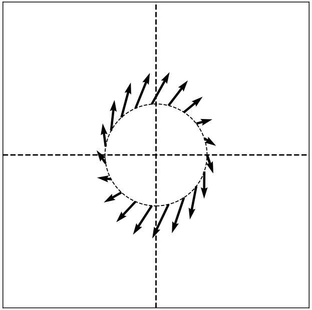
- Because of linearity, along any given direction $\vec{f}$ changes only in scale.

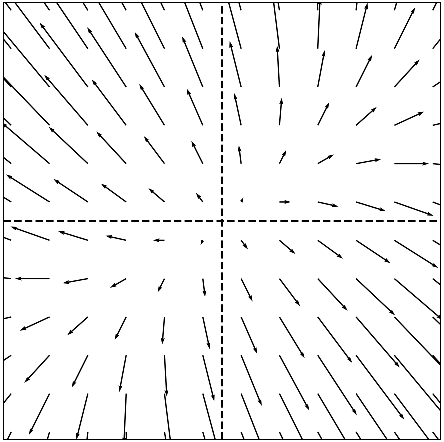
$$\vec{f}(\lambda\hat{x}) = \lambda\vec{f}(\hat{x})$$

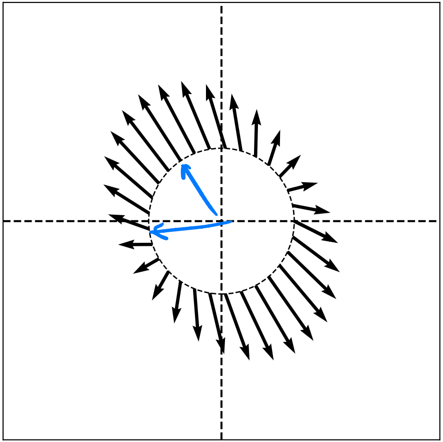
$f(\vec{x})$

$f(2\vec{x}) = 2f(\vec{x})$







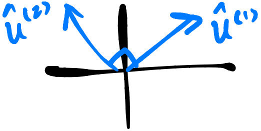


Linear Transformations and Bases

- We have been writing transformations in coordinate form. For example:

$$\vec{f}(\vec{x}) = (x_1 + x_2, x_1 - x_2)^T$$

- To do so, we assumed the **standard basis**.
- If we use a different basis, the formula for $\vec{f}$ changes.



Example

2 0

- Suppose that in the standard basis, $\vec{f}(\vec{x}) = (x_1 + x_2, x_1 - x_2)^T$.
- Let $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1, 1)^T$ and $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(-1, 1)^T$.
- Write $[\vec{x}]_{\mathcal{U}} = (z_1, z_2)^T$. $\rightarrow$ $\vec{x} = z_1\hat{u}^{(1)} + z_2\hat{u}^{(2)}$
- What is $[\vec{f}(\vec{x})]_{\mathcal{U}}$ in terms of z_1 and z_2?

$$\vec{f}(\vec{x}) = \alpha\hat{u}^{(1)} + \beta\hat{u}^{(2)}$$

$$\vec{f}(\hat{u}^{(1)}) = \vec{f}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\vec{f}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}2\\0\end{pmatrix} = \frac{2}{\sqrt{2}}\hat{e}^{(1)}$$

$$\vec{f}(\hat{u}^{(2)}) = \vec{f}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\vec{f}\begin{pmatrix}-1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-2\end{pmatrix} = \frac{-2}{\sqrt{2}}\hat{e}^{(2)}$$

Example

- Suppose that in the standard basis, $\vec{f}(\vec{x}) = (x_1 + x_2, x_1 - x_2)^T$.
- Let $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1, 1)^T$ and $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(-1, 1)^T$.
- Write $[\vec{x}]_{\mathcal{U}} = (z_1, z_2)^T$.
- What is $[\vec{f}(\vec{x})]_{\mathcal{U}}$ in terms of z_1 and z_2?

$$\vec{f}(\vec{u}^{(1)}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 \\ 0 \end{pmatrix} \qquad [\vec{f}(\vec{u}^{(1)})]_{\mathcal{U}} = \begin{pmatrix} \vec{f}(\hat{u}^{(1)}) \cdot \hat{u}^{(1)} \\ \vec{f}(\hat{u}^{(1)}) \cdot \hat{u}^{(2)} \end{pmatrix}$$

$$\vec{f}(\hat{u}^{(1)}) \cdot \hat{u}^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} 2 = 1$$

$$\vec{f}(\hat{u}^{(1)}) \cdot \hat{u}^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{2}(-2) = -1$$

$$[\vec{f}(\hat{u}^{(1)})]_{\mathcal{U}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Longleftrightarrow \vec{f}(\hat{u}^{(1)}) = \hat{u}^{(1)} - \hat{u}^{(2)}$$

Example

- Suppose that in the standard basis, $\vec{f}(\vec{x}) = (x_1 + x_2, x_1 - x_2)^T$.
- Let $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1, 1)^T$ and $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(-1, 1)^T$.
- Write $[\vec{x}]_{\mathcal{U}} = (z_1, z_2)^T$.
- What is $[\vec{f}(\vec{x})]_{\mathcal{U}}$ in terms of z_1 and z_2?

$$[f(\hat{u}^{(2)})]_u \qquad \vec{f}(\hat{u}^{(2)}) = \vec{f}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\vec{f}\begin{pmatrix}-1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-2\end{pmatrix}$$

$$\vec{f}(\hat{u}^{(2)}) \cdot \hat{u}^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-2\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2} \cdot -2 = -1$$

$$\vec{f}(\hat{u}^{(2)}) \cdot \hat{u}^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-2\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix} = \frac{1}{2} \cdot -2 = -1$$

$$[f(\hat{u}^{(2)})]_u = \begin{pmatrix}-1\\-1\end{pmatrix} \iff f(\hat{u}^{(2)}) = -\hat{u}^{(1)} - \hat{u}^{(2)}$$

$\vec{f}(\hat{u}^{(1)}) = \hat{u}^{(1)} - \hat{u}^{(2)}$

$\vec{f}(\hat{u}^{(2)}) = -\hat{u}^{(1)} - \hat{u}^{(2)}$

Example

- Suppose that in the standard basis, $\vec{f}(\vec{x}) = (x_1 + x_2, x_1 - x_2)^T$.
- Let $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1, 1)^T$ and $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(-1, 1)^T$.
- Write $[\vec{x}]_{\mathcal{U}} = (z_1, z_2)^T$. $\vec{x} = z_1 \hat{u}^{(1)} + z_2 \hat{u}^{(2)}$
- What is $[\vec{f}(\vec{x})]_{\mathcal{U}}$ in terms of z_1 and z_2?

$$\vec{f}(\vec{x}) = z_1 \vec{f}(\hat{u}^{(1)}) + z_2 f(\hat{u}^{(2)})$$

$$= z_1 \left[\hat{u}^{(1)} - \hat{u}^{(2)}\right] + z_2 \left[-\hat{u}^{(1)} - \hat{u}^{(2)}\right]$$

$$= (z_1 - z_2)\hat{u}^{(1)} + (-z_1 - z_2)\hat{u}^{(2)}$$

$$[\vec{f}(\vec{x})]_{u} = \begin{pmatrix} z_1 - z_2 \\ -z_1 - z_2 \end{pmatrix}$$

DSC 140B

Representation Learning

Lecture 03 | Part 2

Matrices

Matrices?

- I thought this week was supposed to be about linear algebra... Where are the matrices?

Matrices?

- I thought this week was supposed to be about linear algebra... Where are the matrices?
- What is a matrix, anyways?

What is a matrix?

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Recall: Linear Transformations

- A **transformation** $\vec{f}(\vec{x})$ is a function which takes a vector as input and returns a vector of the same dimensionality.

- A transformation $\vec{f}$ is **linear** if

$$\vec{f}(\alpha\vec{u} + \beta\vec{v}) = \alpha\vec{f}(\vec{u}) + \beta\vec{f}(\vec{v})$$

Recall: Linear Transformations

- **Key** consequence of **linearity**: to compute $\vec{f}(\vec{x})$, only need to know what $\vec{f}$ does to basis vectors.

- Example:

$$\vec{x} = 3\hat{e}^{(1)} - 4\hat{e}^{(2)} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$\vec{f}(\hat{e}^{(1)}) = -\hat{e}^{(1)} + 3\hat{e}^{(2)} = (-1, 3)^T$$

$$\vec{f}(\hat{e}^{(2)}) = 2\hat{e}^{(1)} = (2, 0)^T$$

$$\vec{f}(\vec{x}) = \vec{f}\begin{pmatrix} 3 \\ -4 \end{pmatrix} = 3f(\hat{e}^{(1)}) - 4f(\hat{e}^{(2)}) = 3\begin{pmatrix} -1 \\ 3 \end{pmatrix} - 4\begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 9 \end{pmatrix} + \begin{pmatrix} -8 \\ 0 \end{pmatrix} = \begin{pmatrix} -11 \\ 9 \end{pmatrix}$$

Matrices

- **Idea**: Since $\vec{f}$ is defined by what it does to basis, place $\vec{f}(\hat{e}^{(1)})$, $\vec{f}(\hat{e}^{(2)})$, ... into a table as columns
- This is the **matrix** representing[2] $\vec{f}$

$$\vec{f}(\hat{e}^{(1)}) = -\hat{e}^{(1)} + 3\hat{e}^{(2)} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\vec{f}(\hat{e}^{(2)}) = 2\hat{e}^{(1)} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}$$

---

[2] with respect to the standard basis $\hat{e}^{(1)}, \hat{e}^{(2)}$

Exercise

Write the matrix representing $\vec{f}$ with respect to the standard basis, given:

$$\vec{f}(\hat{e}^{(1)}) = (1, 4, 7)^T$$
$$\vec{f}(\hat{e}^{(2)}) = (2, 5, 7)^T$$
$$\vec{f}(\hat{e}^{(3)}) = (3, 6, 9)^T$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 7 & 9 \end{pmatrix}$$

$$\begin{pmatrix}-2\\-8\\-14\end{pmatrix}+\begin{pmatrix}2\\5\\8\end{pmatrix}+\begin{pmatrix}9\\18\\27\end{pmatrix} = \begin{pmatrix}9\\15\\21\end{pmatrix}$$

Exercise

Suppose $\vec{f}$ has the matrix below:

$$\begin{array}{c} \begin{array}{ccc} f(\hat{e}^{(1)}) & & f(\hat{e}^{(3)}) \end{array} \\ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \\ \begin{array}{ccc} & f(\hat{e}^{(2)}) & \end{array} \end{array}$$

Let $\vec{x} = (-2, 1, 3)^T$. What is $\vec{f}(\vec{x})$?

$$f(\vec{x}) = f\begin{pmatrix}-2\\1\\3\end{pmatrix} = -2\,f(\hat{e}^{(1)}) + 1\,f(\hat{e}^{(2)}) + 3\,f(\hat{e}^{(3)})$$

$$= -2\begin{pmatrix}1\\4\\7\end{pmatrix} + 1\begin{pmatrix}2\\5\\8\end{pmatrix} + 3\begin{pmatrix}3\\6\\9\end{pmatrix} = \begin{pmatrix}-2\\-8\\-14\end{pmatrix}+\begin{pmatrix}2\\5\\8\end{pmatrix}+\begin{pmatrix}9\\18\\27\end{pmatrix}$$

Main Idea

A square $(n \times n)$ matrix can be interpreted as a compact representation of a linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$.

$$AB \qquad f_A(f_B(\vec{x}))$$

$$\begin{array}{cc} \uparrow & \uparrow \\ f_A & f_B \end{array}$$

What is matrix multiplication?

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}}_{\vec{f}} \underbrace{\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 9 \\ 15 \\ 21 \end{pmatrix}}_{\vec{f}(\vec{x})}$$

$$(1)(-2) + (2)(1) + (3)(3) = -2 + 2 + 9 = 9$$

$$-14 + 8 + 27$$

A low-level definition

$$(A\vec{x})_i = \sum_{j=1}^{n} A_{ij} x_j$$

A low-level interpretation

$$\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\overset{\vec{x}}{\begin{pmatrix}-2\\1\\3\end{pmatrix}} = \underset{x_1}{-2}\overset{f(\hat{e}^{(1)})}{\begin{pmatrix}1\\4\\7\end{pmatrix}} + \underset{x_2}{1}\overset{f(\hat{e}^{(2)})}{\begin{pmatrix}2\\5\\8\end{pmatrix}} + \underset{x_3}{3}\overset{f(\hat{e}^{(3)})}{\begin{pmatrix}3\\6\\9\end{pmatrix}}$$

In general...

$$\begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \vec{a}^{(1)} & \vec{a}^{(2)} & \vec{a}^{(3)} \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \vec{a}^{(1)} + x_2 \vec{a}^{(2)} + x_3 \vec{a}^{(3)}$$

Matrix Multiplication

$$\vec{x} = x_1 \hat{e}^{(1)} + x_2 \hat{e}^{(2)} + x_3 \hat{e}^{(3)} = (x_1, x_2, x_3)^T$$

$$\vec{f}(\vec{x}) = x_1 \vec{f}(\hat{e}^{(1)}) + x_2 \vec{f}(\hat{e}^{(2)}) + x_3 \vec{f}(\hat{e}^{(3)})$$

$$A = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \vec{f}(\hat{e}^{(1)}) & \vec{f}(\hat{e}^{(2)}) & \vec{f}(\hat{e}^{(3)}) \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

$$A\vec{x} = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \vec{f}(\hat{e}^{(1)}) & \vec{f}(\hat{e}^{(2)}) & \vec{f}(\hat{e}^{(3)}) \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= x_1 \vec{f}(\hat{e}^{(1)}) + x_2 \vec{f}(\hat{e}^{(2)}) + x_3 \vec{f}(\hat{e}^{(3)})$$

$(X^T X)^{-1} X^T \vec{b}$

Matrix Multiplication

- Matrix A represents a linear transformation $\vec{f}$
  - With respect to the standard basis
  - If we use a different basis, the matrix changes!

- Matrix multiplication $A\vec{x}$ **evaluates** $\vec{f}(\vec{x})$

What are they, *really*?

- Matrices are sometimes just tables of numbers.
- But they often have a deeper meaning.

Main Idea

A square $(n \times n)$ matrix can be interpreted as a compact representation of a linear transformation $\vec{f} : \mathbb{R}^n \to \mathbb{R}^n$.

What's more, if A represents $\vec{f}$, then $A\vec{x} = \vec{f}(\vec{x})$; that is, multiplying by A is the same as evaluating $\vec{f}$.

Example

$$\vec{x} = 3\hat{e}^{(1)} - 4\hat{e}^{(2)} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$\vec{f}(\hat{e}^{(1)}) = -\hat{e}^{(1)} + 3\hat{e}^{(2)} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\vec{f}(\hat{e}^{(2)}) = 2\hat{e}^{(1)} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\vec{f}(\vec{x}) = \begin{pmatrix} -11 \\ 9 \end{pmatrix}$$

$$A = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}$$

$$A\vec{x} = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} -3 - 8 \\ 9 + 0 \end{pmatrix} = \begin{pmatrix} -11 \\ 9 \end{pmatrix}$$

Note

- All of this works because we assumed $\vec{f}$ is **linear**.
- If it isn't, evaluating $\vec{f}$ isn't so simple.

Note

- All of this works because we assumed $\vec{f}$ is **linear**.
- If it isn't, evaluating $\vec{f}$ isn't so simple.
- Linear algebra = simple!

Matrices in Other Bases

- The matrix of a linear transformation wrt the **standard basis**:

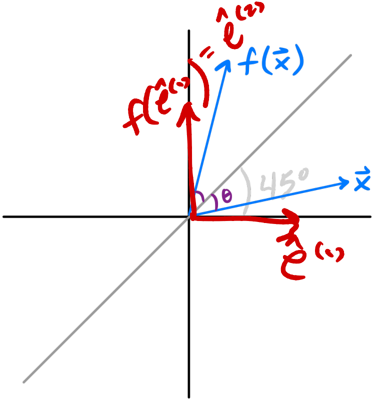
$$\begin{pmatrix} \uparrow & \uparrow & \uparrow & \\ \vec{f}(\hat{e}^{(1)}) & \vec{f}(\hat{e}^{(2)}) & \cdots & \vec{f}(\hat{e}^{(d)}) \\ \downarrow & \downarrow & \downarrow & \end{pmatrix}$$

- With respect to basis $\mathcal{U}$:

$$\begin{pmatrix} \uparrow & \uparrow & \uparrow & \\ [\vec{f}(\hat{u}^{(1)})]_{\mathcal{U}} & [\vec{f}(\hat{u}^{(2)})]_{\mathcal{U}} & \cdots & [\vec{f}(\hat{u}^{(d)})]_{\mathcal{U}} \\ \downarrow & \downarrow & \downarrow & \end{pmatrix}$$

Matrices in Other Bases

- Consider the transformation $\vec{f}$ which "mirrors" a vector over the line of 45°.



$$f(\hat{e}^{(1)}) = \hat{e}^{(2)}$$

$$f(\hat{e}^{(2)}) = \hat{e}^{(1)}$$

- What is its matrix in the standard basis?

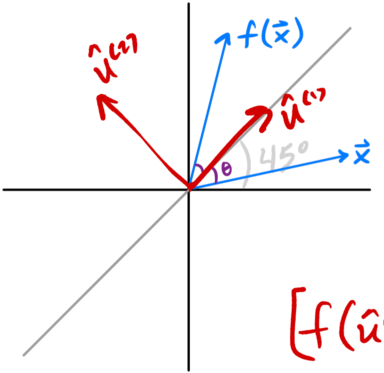
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Matrices in Other Bases

$$f(\hat{u}^{(1)}) = \hat{u}^{(1)}$$

$$f(\hat{u}^{(2)}) = -\hat{u}^{(2)}$$

- Let $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1, 1)^T$
- Let $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(-1, 1)^T$
- What is $[\vec{f}(\hat{u}^{(1)})]_{\mathcal{U}}$?
- $[\vec{f}(\hat{u}^{(2)})]_{\mathcal{U}}$?
- What is the matrix?



$$[f(\hat{u}^{(1)})]_{\mathcal{U}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$[f(\hat{u}^{(2)})]_{\mathcal{U}} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

DSC 140B
Representation Learning

Lecture 03 | Part 3

The Spectral Theorem

Eigenvectors

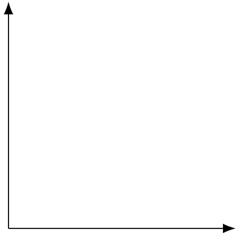
- Let A be an $n \times n$ matrix. An **eigenvector** of A with **eigenvalue** λ is a nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$.

Eigenvectors (of Linear Transformations)

- Let $\vec{f}$ be a linear transformation. An **eigenvector** of $\vec{f}$ with **eigenvalue** λ is a nonzero vector $\vec{v}$ such that $f(\vec{v}) = \lambda\vec{v}$.

Geometric Interpretation

- When $\vec{f}$ is applied to one of its eigenvectors, $\vec{f}$ simply scales it.
  - Possibly by a negative amount.



Symmetric Matrices

- Recall: a matrix A is **symmetric** if $A^T = A$.

The Spectral Theorem[3]

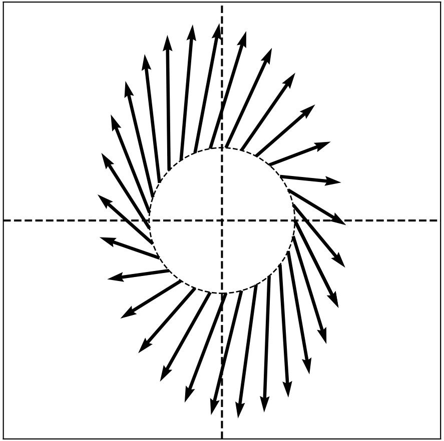
- **Theorem**: Let A be an $n \times n$ *symmetric* matrix. Then there exist n eigenvectors of A which are all mutually orthogonal.

[3] for symmetric matrices

What?

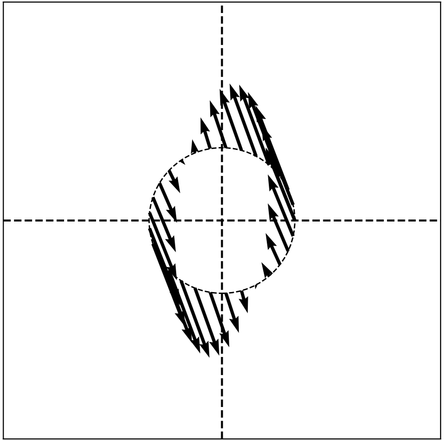
- What does the spectral theorem mean?
- What is an eigenvector, really?
- Why are they useful?

Example Linear Transformation



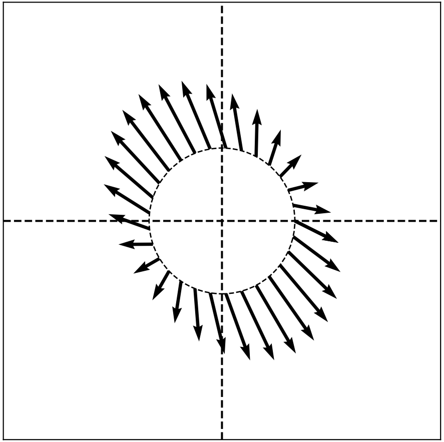
$$A = \begin{pmatrix} 5 & 5 \\ -10 & 12 \end{pmatrix}$$

Example Linear Transformation



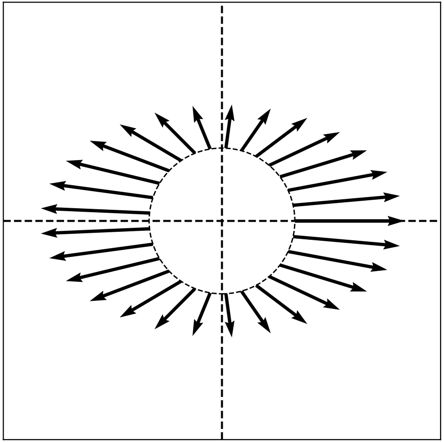
$$A = \begin{pmatrix} -2 & -1 \\ -5 & 3 \end{pmatrix}$$

Example Symmetric Linear Transformation



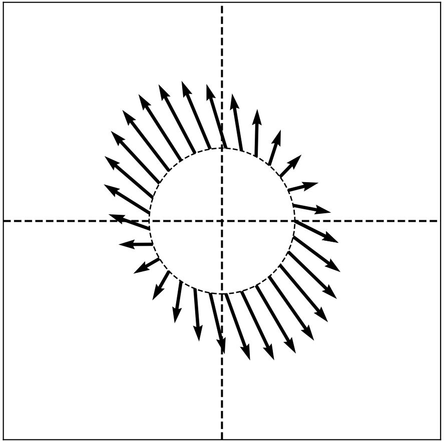
$$A = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

Example Symmetric Linear Transformation



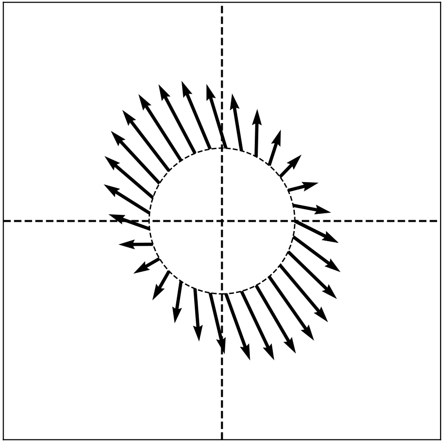
$$A = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$$

Observation #1



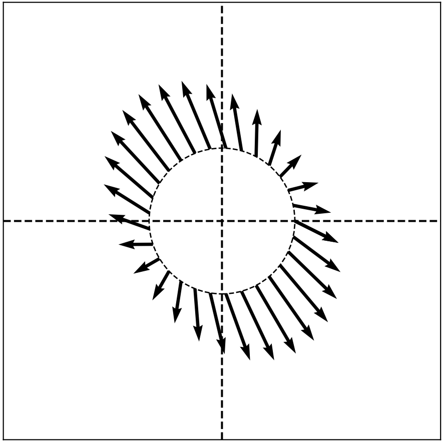
- Symmetric linear transformations have **axes of symmetry**.

Observation #2



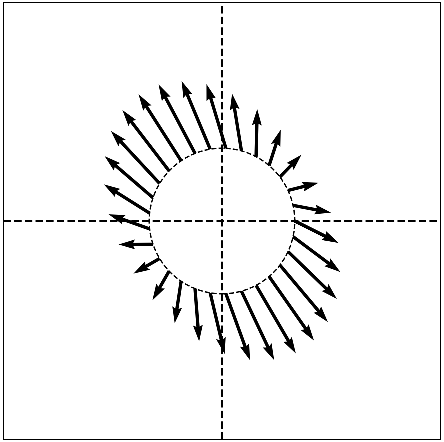
- The axes of symmetry are **orthogonal** to one another.

Observation #3



- The action of $\vec{f}$ along an axis of symmetry is simply to scale its input.

Observation #4



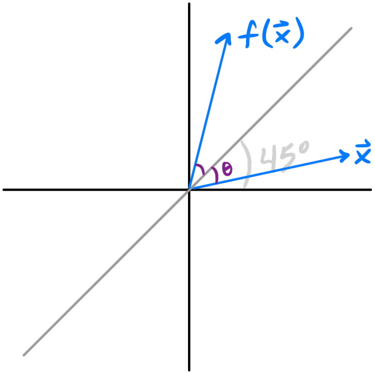
- The size of this scaling can be different for each axis.

Main Idea

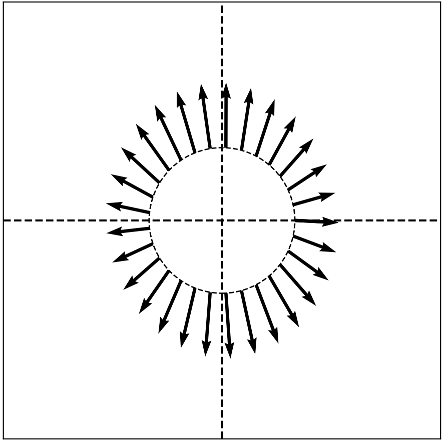
The **eigenvectors** of a symmetric linear transformation (matrix) are its axes of symmetry. The **eigenvalues** describe how much each axis of symmetry is scaled.

Exercise

Consider the linear transformation which mirrors its input over the line of 45°. Give two orthogonal eigenvector of the transformation.

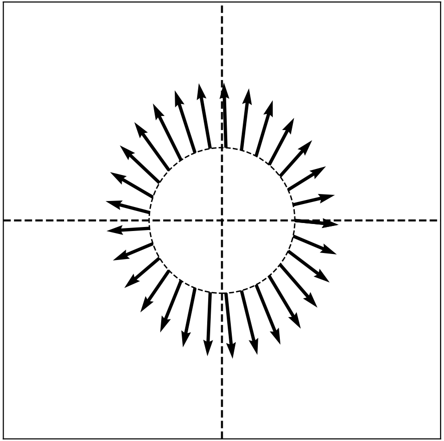


Off-diagonal elements



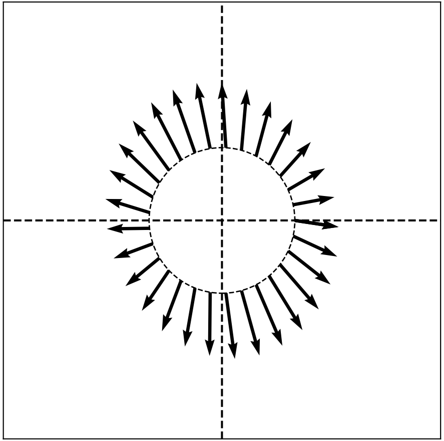
$$A = \begin{pmatrix} 5 & -0.1 \\ -0.1 & 2 \end{pmatrix}$$

Off-diagonal elements



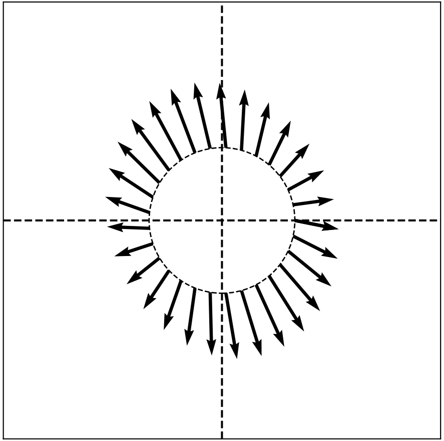
$$A = \begin{pmatrix} 5 & -0.2 \\ -0.2 & 2 \end{pmatrix}$$

Off-diagonal elements



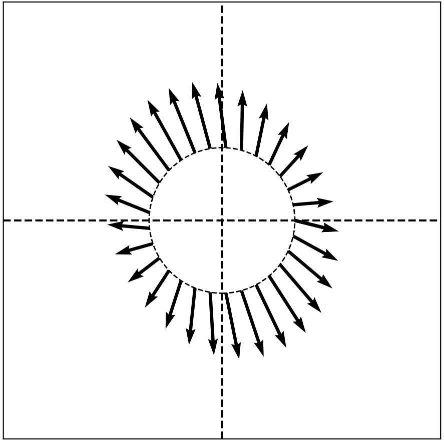
$$A = \begin{pmatrix} 5 & -0.3 \\ -0.3 & 2 \end{pmatrix}$$

Off-diagonal elements



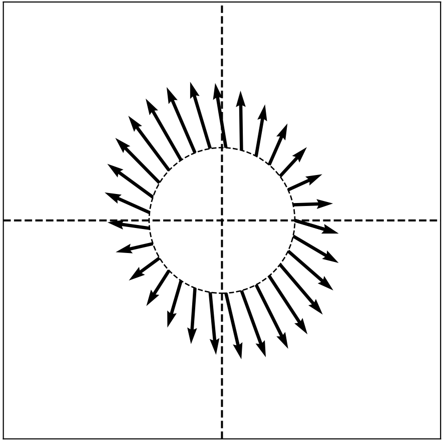
$$A = \begin{pmatrix} 5 & -0.4 \\ -0.4 & 2 \end{pmatrix}$$

Off-diagonal elements



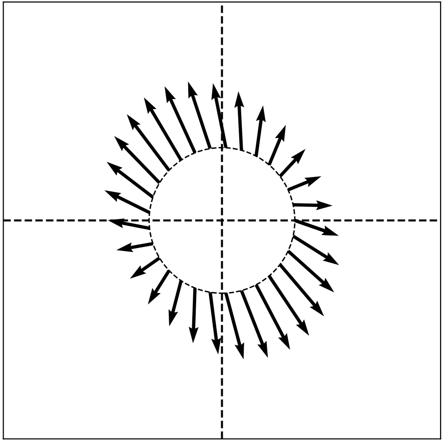
$$A = \begin{pmatrix} 5 & -0.5 \\ -0.5 & 2 \end{pmatrix}$$

Off-diagonal elements



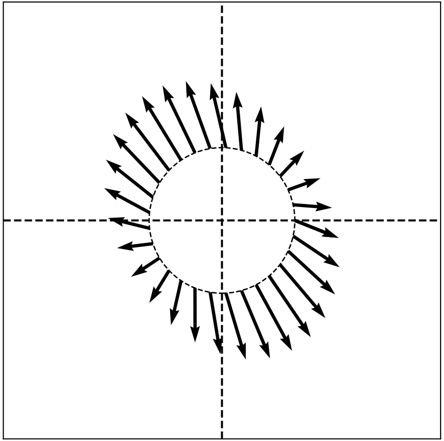
$$A = \begin{pmatrix} 5 & -0.6 \\ -0.6 & 2 \end{pmatrix}$$

Off-diagonal elements



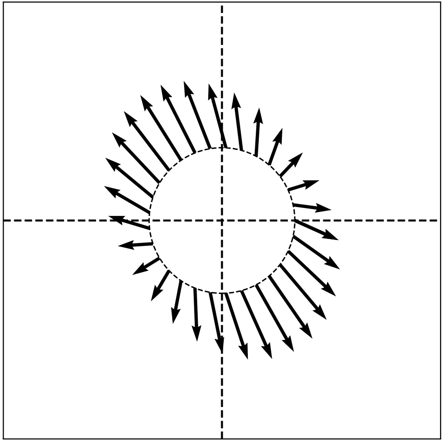
$$A = \begin{pmatrix} 5 & -0.7 \\ -0.7 & 2 \end{pmatrix}$$

Off-diagonal elements



$$A = \begin{pmatrix} 5 & -0.8 \\ -0.8 & 2 \end{pmatrix}$$

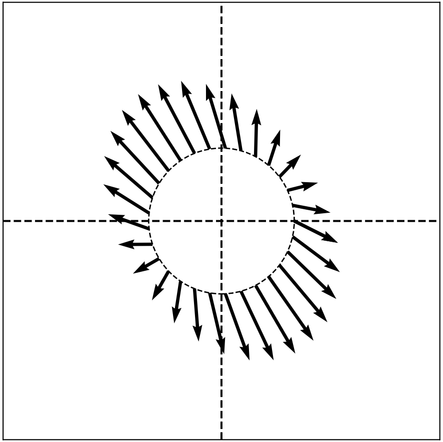
Off-diagonal elements



$$A = \begin{pmatrix} 5 & -0.9 \\ -0.9 & 2 \end{pmatrix}$$

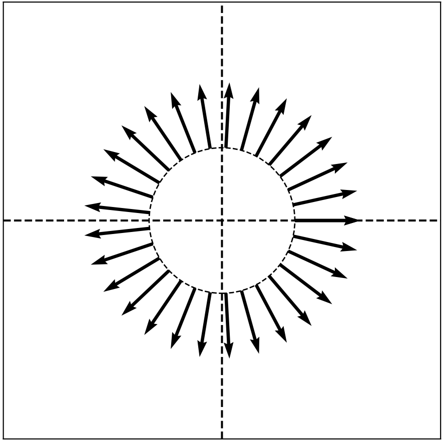
The Spectral Theorem[4]

- **Theorem**: Let A be an $n \times n$ *symmetric* matrix. Then there exist n eigenvectors of A which are all mutually orthogonal.



[4]for symmetric matrices

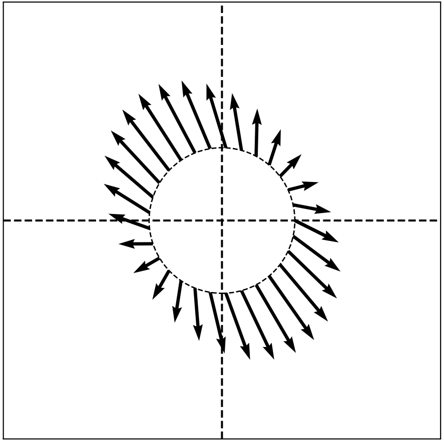
What about total symmetry?



- Every vector is an eigenvector.

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

Computing Eigenvectors



```
»> A = np.array([[2, -1], [-1, 3]])
»> np.linalg.eigh(A)
(array([1.38196601, 3.61803399]),
 array([[-0.85065081, -0.52573111],
        [-0.52573111,  0.85065081]]))
```