DSC 140B
Representation Learning

Lecture 11 | Part 1

Linear Limitations

Linear Predictors

- Last time, we saw linear prediction functions:

$$\begin{aligned} H(\vec{x}; \vec{w}) &= w_0 + w_1 x_1 + \ldots + w_d x_d \\ &= \text{Aug}(\vec{x}) \cdot \vec{w} \end{aligned}$$

Linear Decision Functions

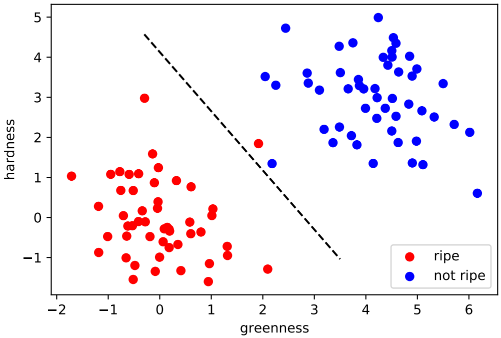
- A linear prediction function H outputs a number.
- What if classes are +1 and -1?
- Can be turned into a **decision function** by taking:

$$\operatorname{sign}(H(\vec{x}))$$

- **Decision boundary** is where $H = 0$
  - Where the sign switches from positive to negative.

Decision Boundaries

- A linear decision function's decision boundary is linear.
  - A line, plane, hyperplane, etc.



An Example: Parking Predictor

- **Task**: Predict (yes / no): Is there parking available at UCSD right now?

- What training data to collect? What features?

Useful Features

- Time of day?
- Day’s high temperature?
- ...

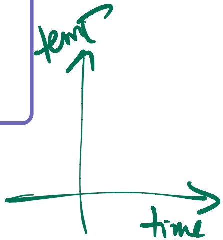
Exercise

Imagine a scatter plot of the training data with the two features:

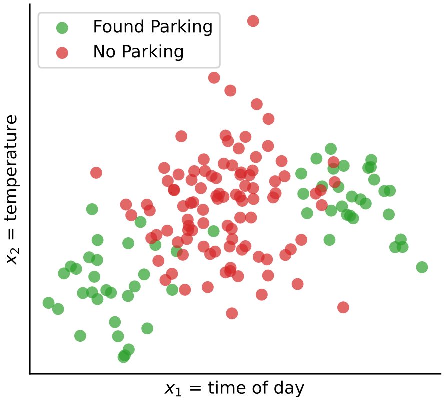
- x_1 = time of day
- x_2 = temperature

"yes" examples are green, "no" are red.

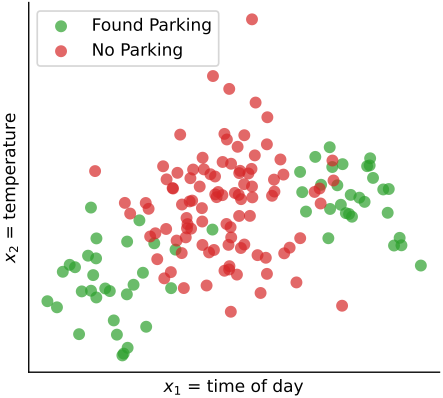
What does it look like?



Parking Data



Uh oh



- A linear decision function won't work.
- What do we do?

Today’s Question

- How do we learn non-linear patterns using linear prediction functions?

DSC 140B

Representation Learning

Lecture 11 | Part 2

Feature Maps

Representations

- We **represented** the data with two features: time and temperature

- In this **representation**, the trend is **nonlinear**.
  - There is no good linear decision function
  - Learning is “difficult”.

Idea

- **Idea**: We'll make a new **representation** by creating **new features** from the **old features**.
- The "right" representation makes the problem easy again.
- What new features should we create?

New Feature Representation

- Linear prediction functions[1] work well when relationship is linear
  - When x is small we should predict -1
  - When x is large we should predict +1
- But parking's relationship with time is not linear:
  - When time is small we should predict +1
  - When time is medium we should predict -1
  - When time is large we should predict +1

[1] Remember: they are weighted votes.

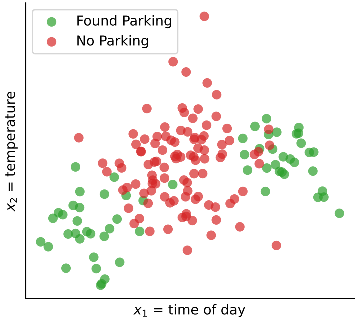
Exercise

How can we "transform" the time of day x_1 to create a new feature x_1' satisfying:

- When x_1' is small, we should predict -1
- When x_1' is large, we should predict +1

What about the temperature, x_2?

Idea



- Transform “time” to “absolute time until/since Noon”
- Transform “temp.” to “absolute difference between temp. and 72°”

Basis Functions

- We will transform:
  1) - the time, x_1, to $|x_1 - \text{Noon}|$
  2) - the temperature, x_2, to $|x_2 - 72^\circ|$
- Formally, we've designed non-linear **basis functions**:

$$\varphi_1(x_1, x_2) = |x_1 - \text{Noon}|$$
$$\varphi_2(x_1, x_2) = |x_2 - 72^\circ|$$

- In general a basis function φ maps $\mathbb{R}^d \to \mathbb{R}$

Feature Mapping

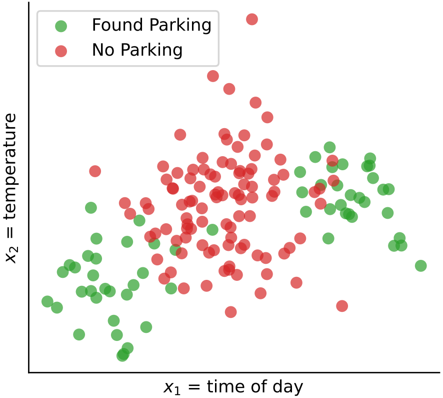
- Define $\vec{\varphi}(\vec{x}) = (\varphi_1(\vec{x}), \varphi_2(\vec{x}))^T$. $\vec{\varphi}$ is a **feature map**
  - Input: vector in “old” representation
  - Output: vector in “new” representation

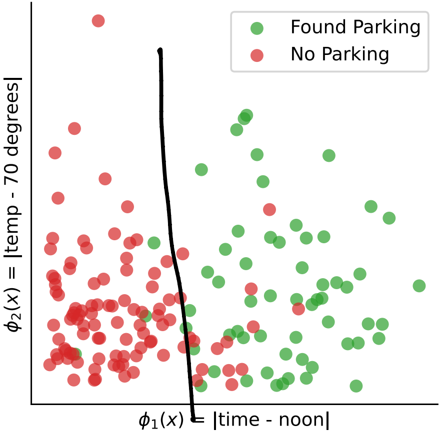
- Example: (2pm 69°)

$$\vec{\varphi}((10\text{a.m.}, 75^\circ)^T) = (2 \text{ hours}, 3^\circ)^T$$

- $\vec{\varphi}$ maps raw data to a **feature space**.

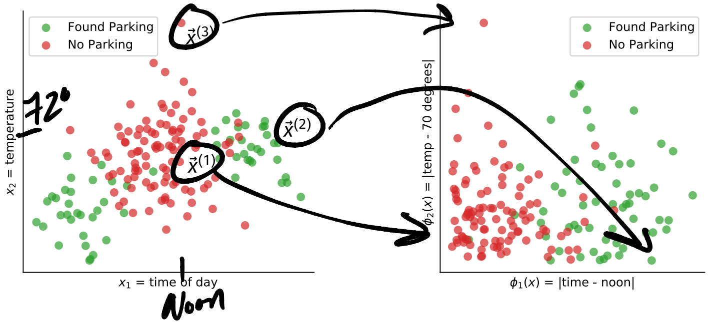
Feature Space, Visualized



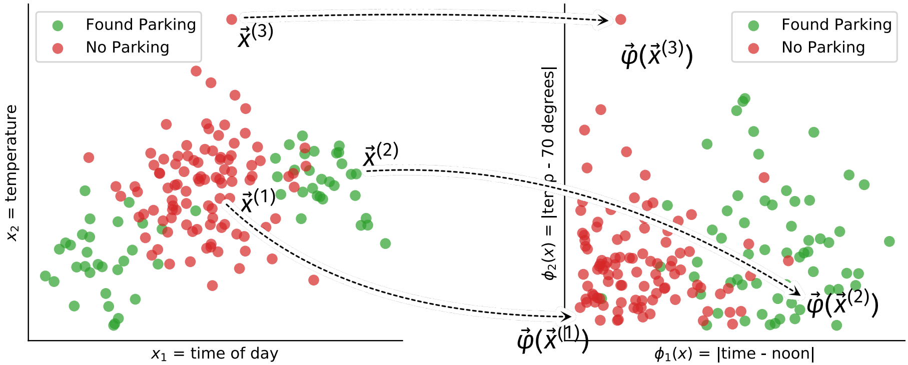


Exercise

Where does $\vec{\varphi}$ map $\vec{x}^{(1)}, \vec{x}^{(2)}$, and $\vec{x}^{(3)}$?



Solution


Found Parking
No Parking
$\vec{x}^{(3)}$
$\vec{x}^{(2)}$
$\vec{x}^{(1)}$
x_2 = temperature
x_1 = time of day
$\vec{\varphi}(\vec{x}^{(3)})$
$\vec{\varphi}(\vec{x}^{(2)})$
$\vec{\varphi}(\vec{x}^{(1)})$
$\phi_2(x)$ = |temp - 70 degrees|
$\phi_1(x)$ = |time - noon|
Found Parking
No Parking

After the Mapping

- The basis functions φ_1, φ_2 give us our “new” features.
- This gives us a new **representation**.
- In this representation, learning (classification) is easier.

Training

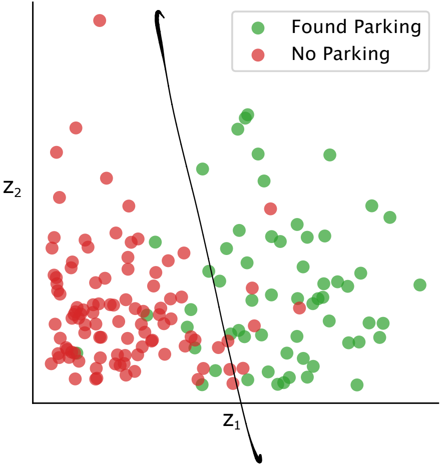
- Map each training example $\vec{x}^{(i)}$ to feature space, creating new training data:

$$\vec{z}^{(1)} = \vec{\varphi}(\vec{x}^{(1)}), \quad \vec{z}^{(2)} = \vec{\varphi}(\vec{x}^{(2)}), \quad \ldots, \quad \vec{z}^{(n)} = \vec{\varphi}(\vec{x}^{(n)})$$

- Fit linear prediction function H in usual way:

$$H_f(\vec{z}) = w_0 + w_1 z_1 + w_2 z_2 + \ldots + w_d z_d$$

Training Data in Feature Space



Prediction

- If we have $\vec{z}$ in feature space, prediction is:

$$H_f(\vec{z}) = w_0 + w_1 z_1 + w_2 z_2 + \ldots + w_d z_d$$

Prediction

- But if we have $\vec{x}$ from original space, we must "convert" $\vec{x}$ to feature space first:

$$
\begin{aligned}
H(\vec{x}) &= H_f(\vec{\varphi}(\vec{x})) \\
&= H_f\big((\overbrace{\varphi_1(\vec{x})}^{z_1}, \overbrace{\varphi_2(\vec{x})}^{z_2}, \ldots, \overbrace{\varphi_d(\vec{x})}^{z_d})^T \big) \\
&= w_0 + w_1 \underbrace{\varphi_1(\vec{x})}_{z_1} + w_2 \underbrace{\varphi_2(\vec{x})}_{z_2} + \ldots + w_d \underbrace{\varphi_d(\vec{x})}_{z_d}
\end{aligned}
$$

Overview: Feature Mapping

- A basis function can involve any/all of the original features:

$$\varphi_3(\vec{x}) = x_1 \cdot x_2$$

- We can make more basis functions than original features:

$$\vec{\varphi}(\vec{x}) = (\,\varphi_1(\vec{x}),\ \varphi_2(\vec{x}),\ \varphi_3(\vec{x})\,)^T$$

Overview: Feature Mapping

1. Start with data in original space, $\mathbb{R}^d$.

2. Choose some basis functions, $\varphi_1, \varphi_2, \ldots, \varphi_{d'}$

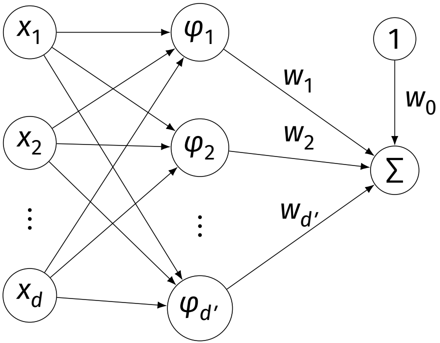
3. Map each data point to **feature space** $\mathbb{R}^{d'}$:

$$\vec{x} \mapsto (\varphi_1(\vec{x}), \varphi_2(\vec{x}), \ldots, \varphi_{d'}(\vec{x}))^t$$

4. Fit linear prediction function in new space:

$$H(\vec{x}) = w_0 + w_1\varphi_1(\vec{x}) + w_2\varphi_2(\vec{x})$$

$$H(\vec{x}) = w_0 + w_1\varphi_1(\vec{x}) + w_2\varphi_2(\vec{x})$$



Today's Question

- Q: How do we learn non-linear patterns using linear prediction functions?
- A: Use non-linear basis functions to map to a feature space.

DSC 140B
Representation Learning

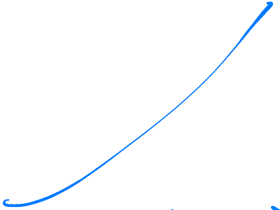
Lecture 11 | Part 3

Basis Functions and Regression

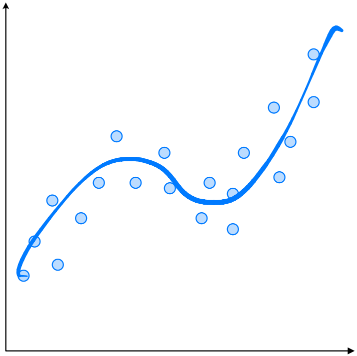
By the way...

- You’ve (probably) seen basis functions used before.
- Linear regression for non-linear patterns in DSC 40A.

Example



$\varphi(x) = e^x$



Fitting Non-Linear Patterns

- Fit function of the form

$$H(x) = w_0 + w_1 x + w_2 x^2 + w_3 x^3 + w_4 x^4$$

- Linear function of $\vec{w}$, non-linear function of x.

The Trick

- Treat x, x^2, x^3, x^4 as **new** features.
- Create design matrix:

$$X = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & x_n^3 & x_n^4 \end{pmatrix}$$

- Solve $X^T X \vec{w} = X^T \vec{w}$ for $\vec{w}$, as usual.
- Works for more than just polynomials.

Another View

- We have changed the representation of a point:

$$x \mapsto (x, x^2, x^3, x^4)$$

- Basis functions:

$$\varphi_1(x) = x \quad \varphi_2(x) = x^2 \quad \varphi_3(x) = x^3 \quad \varphi_4(x) = x^4$$

DSC 140B
Representation Learning

Lecture 11 | Part 4

A Tale of Two Spaces

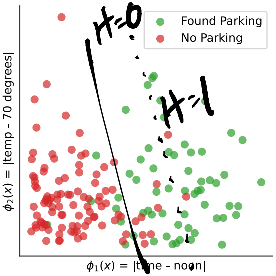
A Tale of Two Spaces

- The **original space**: where the raw data lies.
- The **feature space**: where the data lies after feature mapping $\vec{\varphi}$
- Remember: we fit a linear prediction function in the **feature space**.

Exercise

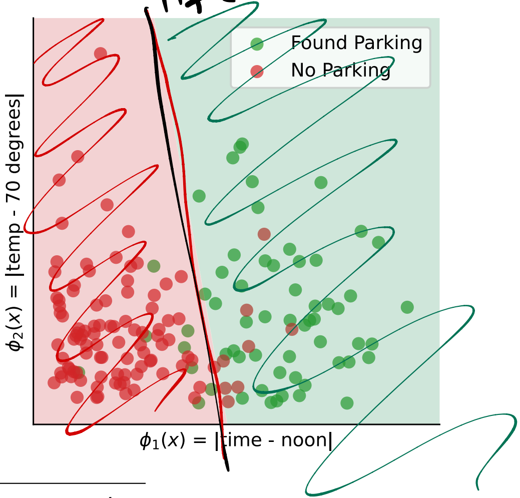
H

- In **feature space**, what does the decision boundary look like?
- What does the prediction function surface look like?



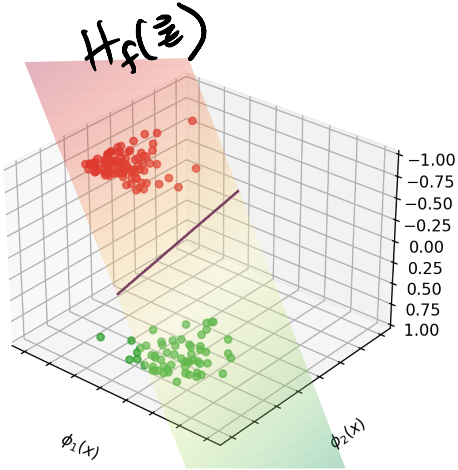
Decision Boundary in Feature Space[2]

$H_f(\vec{z}) = 0$



[2] Fit by minimizing square loss

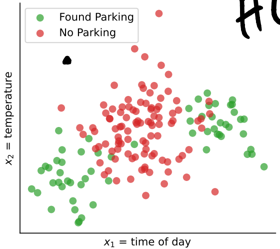
Prediction Surface in Feature Space



Exercise

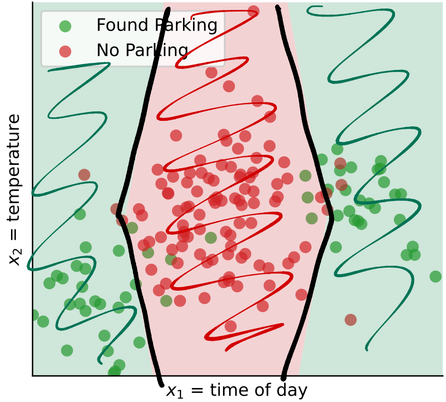
- In the **original space**, what does the decision boundary look like?
- What does the prediction function surface look like?

$$H(\vec{x}) = H_f(\vec{\varphi}(\vec{x}))$$



Decision Boundary in Original Space[3]

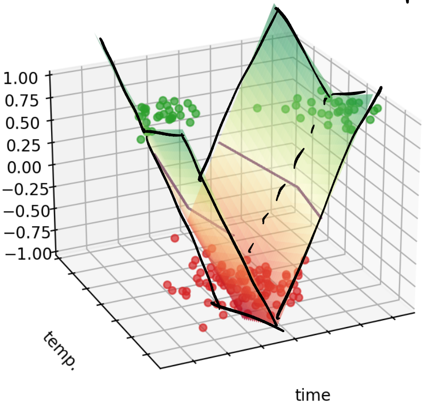
$H(\vec{x}) = 0$



[3] Fit by minimizing square loss

Prediction Surface in Original Space

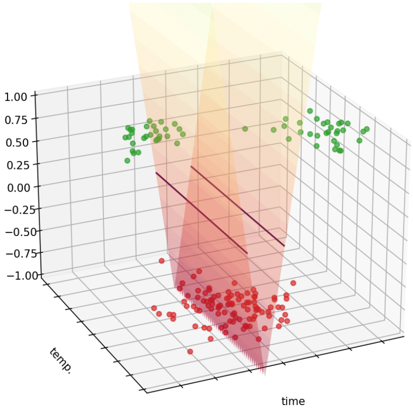
$H(\vec{x})$

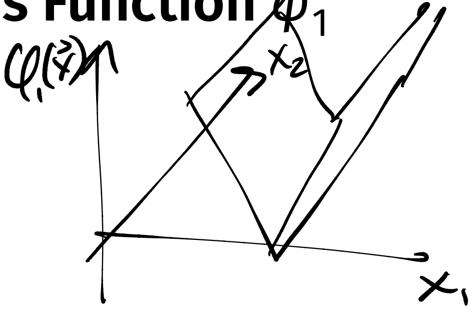


Insight

- H is a sum of basis functions, φ_1 and φ_2.
  - $H(\vec{x}) = w_0 + \underbrace{w_1\varphi_1(\vec{x})}_{f_1(x)} + \underbrace{w_2\varphi_2(\vec{x})}_{f_2(x)}$
- The prediction surface is a sum of other surfaces.
- Each basis function is a “building block”.

Visualizing the Basis Function φ_1

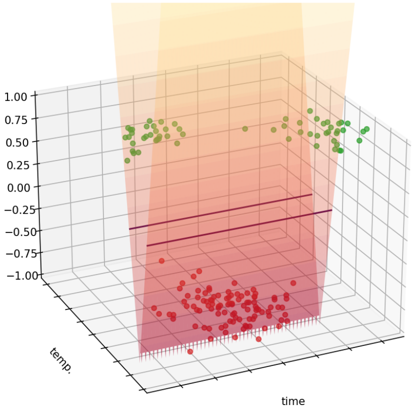




- $w_0 + w_1 |x_1 - \text{noon}|$

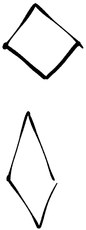
$\varphi_1(\vec{x})$

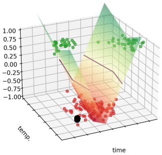
Visualizing the Basis Function φ_2



- $w_0 + w_2 |x_2 - 72^\circ|$

Visualizing the Prediction Surface





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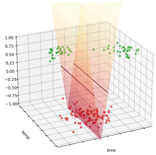

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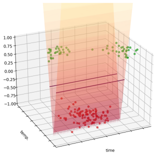
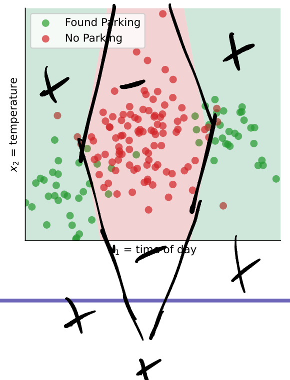


Exercise

The decision boundary has a single "pocket" where it is negative. Can it have more than one, assuming we use basis functions of the same form? What if we use more than two basis functions?

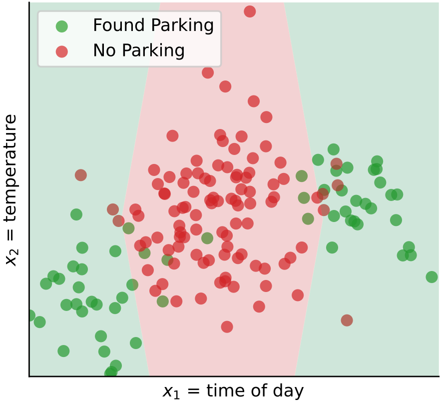


$|x_i - c|$

Answer: No!

- Recall: the sum of **convex** functions is **convex**.
- Each of our basis functions is convex.
- So the prediction surface will be convex, too.
- Limited in what patterns they can classify.

View: Function Approximation



- Find a function that is ≈ 1 near green points and ≈ −1 near red points.

What's Wrong?

- We've discovered how to learn non-linear patterns using linear prediction functions.
  - Use non-linear basis functions to map to a feature space.

- Something should bug you, though…

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Representation Learning

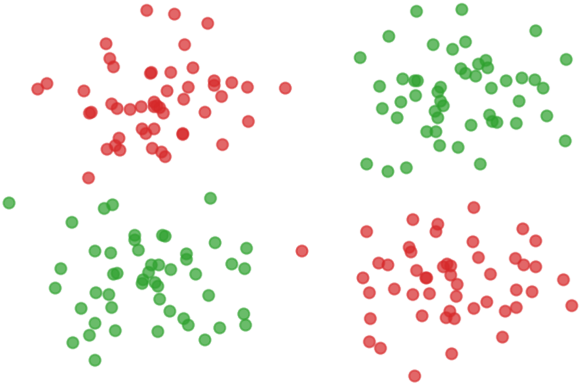
Lecture 11 | Part 5

Radial Basis Functions

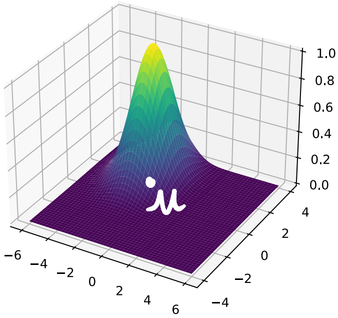
Choosing Basis Functions

- Our previous basis functions have limitations.
- They are convex: prediction surface can only have one negative/positive region.
- They diverge $\to \infty$ away from their centers.
  - They get more “confident”?

Example

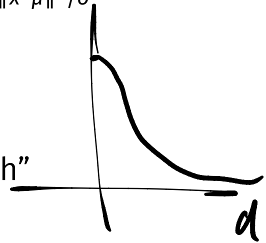


Gaussian Basis Functions



- A common choice: **Gaussian** basis functions:

$$\varphi(\vec{x};\vec{\mu},\sigma) = e^{-\overbrace{\|\vec{x}-\vec{\mu}\|}^{d}{}^2/\sigma^2}$$



- $\vec{\mu}$ is the center.
- σ controls the "width"

Gaussian Basis Function

- If $\vec{x}$ is close to $\vec{\mu}$, $\varphi(\vec{x}; \vec{\mu}, \sigma)$ is large.
- If $\vec{x}$ is far from $\vec{\mu}$, $\varphi(\vec{x}; \vec{\mu}, \sigma)$ is small.
- Intuition: φ measures how “similar” $\vec{x}$ is to $\vec{\mu}$.
  - Assumes that “similar” objects have close feature vectors.

New Representation

- Pick number of new features, d'.
- Pick centers for Gaussians $\vec{\mu}^{(1)}, \ldots, \vec{\mu}^{(2)}, \ldots, \vec{\mu}^{(d')}$
- Pick widths: $\sigma_1, \sigma_2, \ldots, \sigma_{d'}$ (usually all the same)
- Define ith basis function:

$$\varphi_i(\vec{x}) = e^{-\|\vec{x}-\vec{\mu}^{(i)}\|^2/\sigma_i^2}$$

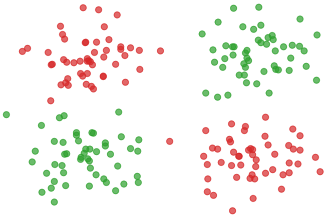
New Representation

- For any feature vector $\vec{x} \in \mathbb{R}^d$, map to vector $\vec{\varphi}(\vec{x}) \in \mathbb{R}^{d'}$.
    - φ_1: "similarity" of $\vec{x}$ to $\vec{\mu}^{(1)}$
    - φ_2: "similarity" of $\vec{x}$ to $\vec{\mu}^{(2)}$
    - ...
    - $\varphi_{d'}$: "similarity" of $\vec{x}$ to $\vec{\mu}^{(d')}$

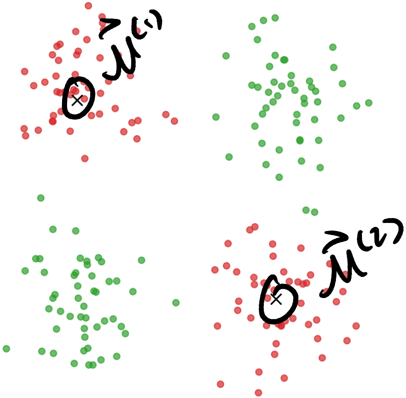
- Train linear classifier in this new representation.
    - E.g., by minimizing expected square loss.

Exercise

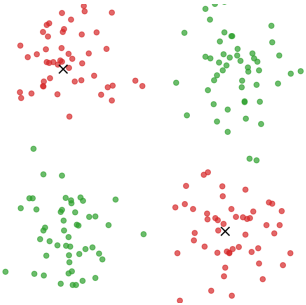
How many Gaussian basis functions would you use, and where would you place them to create a new representation for this data?

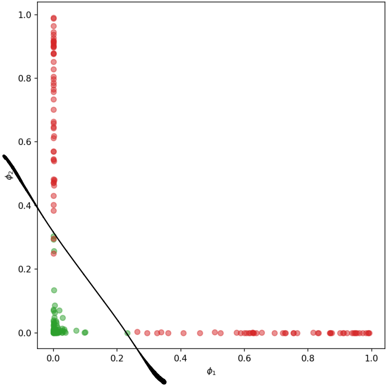


Placement



Feature Space





Prediction Function

- $H(\vec{x})$ is a sum of Gaussians:

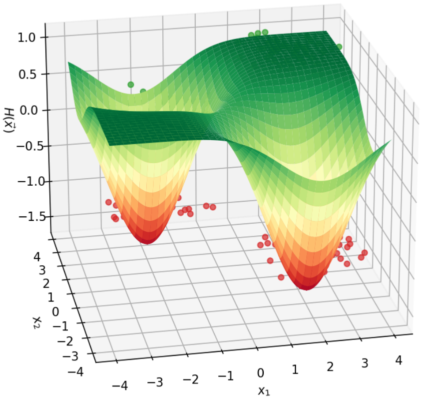
$$
\begin{aligned}
H(\vec{x}) &= w_0 + w_1 \varphi_1(\vec{x}) + w_2 \varphi_2(\vec{x}) + \ldots \\
&= w_0 + w_1 e^{-\|\vec{x}-\vec{\mu}_1\|^2/\sigma^2} + w_2 e^{-\|\vec{x}-\vec{\mu}_2\|^2/\sigma^2} + \ldots
\end{aligned}
$$

Exercise

What does the surface of the prediction function look like?

Hint: what does the sum of 1-d Gaussians look like?

Prediction Function Surface

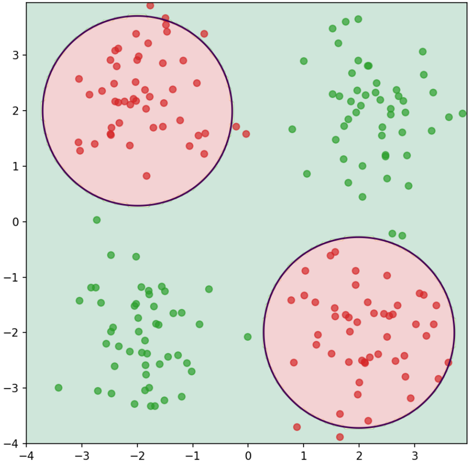


$$H(\vec{x}) = w_0 + w_1 e^{-\|\vec{x}-\vec{\mu}_1\|^2/\sigma^2} + w_2 e^{-\|\vec{x}-\vec{\mu}_2\|^2/\sigma^2}$$

An Interpretation

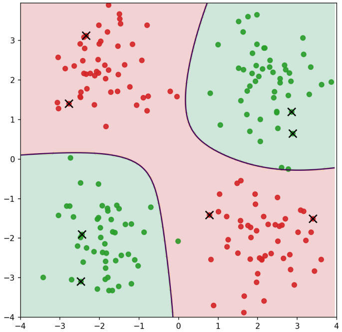
- Basis function φ_i makes a “bump” in surface of H
- w_i adjusts the “prominance” of this bump

Decision Boundary

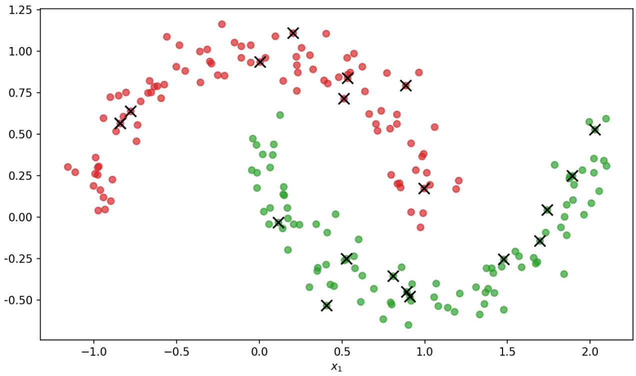


More Features

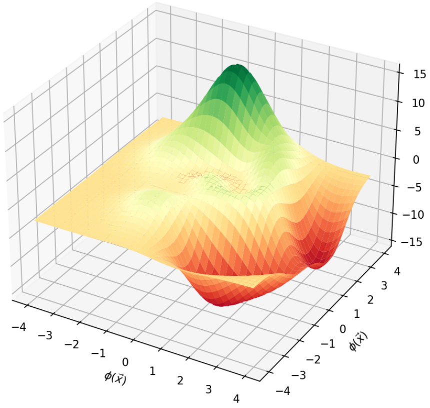
- By increasing number of basis functions, we can make more complex decision surfaces.



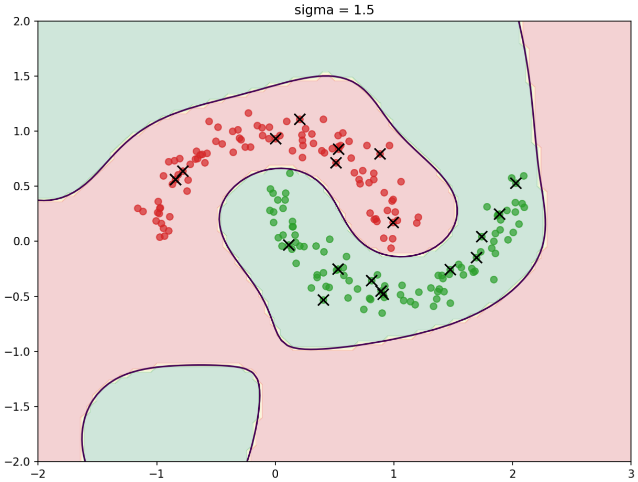
Another Example



Prediction Surface



Decision Boundary



Radial Basis Functions

- Gaussians are examples of **radial basis functions**.
- Each basis function has a **center**, $\vec{c}$.
- Value depends only on distance from center:

$$\varphi(\vec{x}; \vec{c}) = f(\|\vec{x} - \vec{c}\|)$$

Another Radial Basis Function

- **Multiquadric**: $\varphi(\vec{x};\vec{c}) = \sqrt{\sigma^2 + \|\vec{x} - \vec{c}\|}/\sigma$