Lecture 12 – Probability

DSC 10, Fall 2022

Announcements

- Lab 4 is due **Saturday at 11:59PM**.
- Homework 4 is due on **Tuesday 10/25 at 11:59PM**.
- The Midterm Project will be released **today** and is due **Tuesday 11/1.**
    - It takes much longer than a homework, so start now!
    - Partners are optional but recommended, and can be from any lecture section.
    - If you're looking for a partner, email tutor Anna (a2liu@ucsd.edu).
    - You must use the **pair programming** model when working with a partner.

Agenda

We'll cover the basics of probability theory. This is a math lesson; take written notes. ✍🏽

Probability resources

Probability is a tricky subject. If it doesn't click during lecture or on the assignments, take a look at the following resources:

- **Computational and Inferential Thinking, Chapter 9.5**.
- **Theory Meets Data, Chapters 1 and 2**.
- **Khan Academy's unit on Probability**.

Probability theory

Probability theory

- Some things in life *seem* random.
    - e.g. flipping a coin or rolling a die 🎲.
- The **probability** of seeing "heads" when flipping a fair coin is $\frac{1}{2}$.
- One interpretation of probability says that if we flipped a coin infinitely many times, then $\frac{1}{2}$ of the outcomes would be heads.

Terminology

- **Experiment**: A process or action whose result is random.
    - e.g., rolling a die.
    - e.g., flipping a coin twice.
- **Outcome**: The result of an experiment.
    - e.g., the possible outcomes of rolling a six-sided die are 1, 2, 3, 4, 5, and 6.
    - e.g., the possible outcomes of flipping a coin twice are HH, HT, TH, and TT.
- **Event**: A set of outcomes.
    - e.g., the event that the die lands on a even number is the set of outcomes {2, 4, 6}.
    - e.g., the event that the die lands on a 5 is the set of outcomes {5}.
    - e.g., the event that there is at least 1 head in 2 flips is the set of outcomes {HH, HT, TH}.

Terminology

- **Probability**: A number between 0 and 1 (equivalently, between 0% and 100%) which describes the likelihood of an event.
    - 0: the event never happens.
    - 1: the event always happens.
- Notation: if A is an event, $P(A)$ is the probability of that event.

Equally-likely outcomes

- If all outcomes in event A are equally likely, then the probability of A is

$$P(A) = \frac{\#\text{ of outcomes satisfying } A}{\text{total } \#\text{ of outcomes}} = \frac{3}{8}$$

- **Example 1:** Suppose we flip a fair coin 3 times. What is the probability we see exactly 2 heads?

A = see 2 H

incorrect:
- 4 outcomes
  0 heads, 1 head
  2 heads, 3 heads
- prob = 1/4

possible outcomes

8

HHH, HTH, THH, TTH

HHT, HTT, THT, TTT

Concept Check ✅ – Answer at cc.dsc10.com

R B G

I have three cards: red, blue, and green. What is the chance that I choose a card at random and it is green, then – **without putting it back** – I choose another card at random and it is red?

- A) $\frac{1}{9}$
- B) $\frac{1}{6}$
- C) $\frac{1}{3}$
- D) $\frac{2}{3}$
- E) None of the above.

$\frac{1}{3}$ AND $\frac{1}{2}$

$\frac{1}{3} * \frac{1}{2}$

multiplication rule

possible outcomes

6

RB BG GB

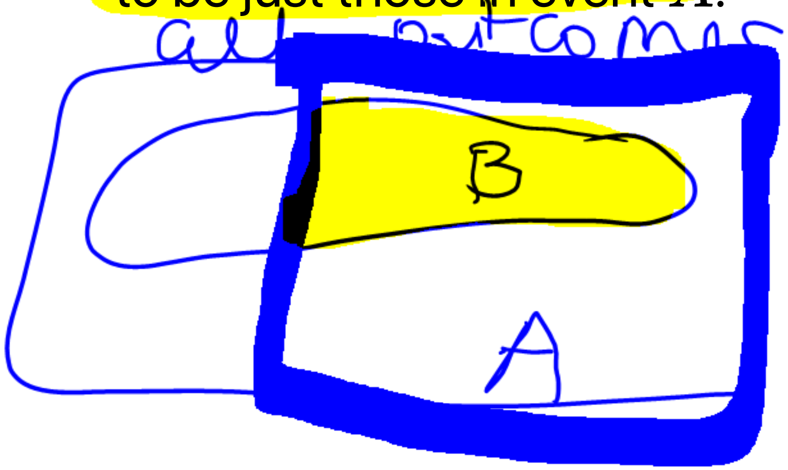
RG BR GR

Conditional probabilities

- Two events A and B can both happen. Suppose that we know A has happened, but we don't know if B has.
- If all outcomes are equally likely, then the conditional probability of B given A is:

$$P(B \text{ given } A) = \frac{\# \text{ of outcomes satisfying both } A \text{ and } B}{\# \text{ of outcomes satisfying } A}$$

- Intuitively, this is similar to the definition of the regular probability of B, $P(B) = \frac{\# \text{ of outcomes satisfying } B}{\text{total } \# \text{ of outcomes}}$, if you restrict the set of possible outcomes to be just those in event A.



Concept Check ✅ – Answer at cc.dsc10.com

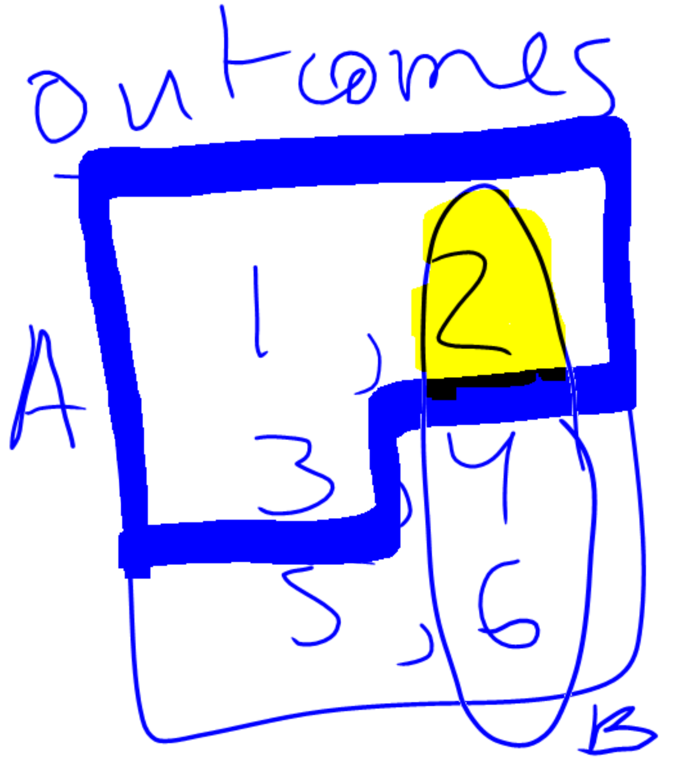
$$P(B \text{ given } A) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\#\text{ of outcomes satisfying } A} = \frac{1}{3}$$

A

I roll a six-sided die and don't tell you what the result is, but I tell you that it is 3 or less. What is the probability that the result is even?

B

- A) $\frac{1}{2}$
- B) $\frac{1}{3}$
- C) $\frac{1}{4}$
- D) None of the above.



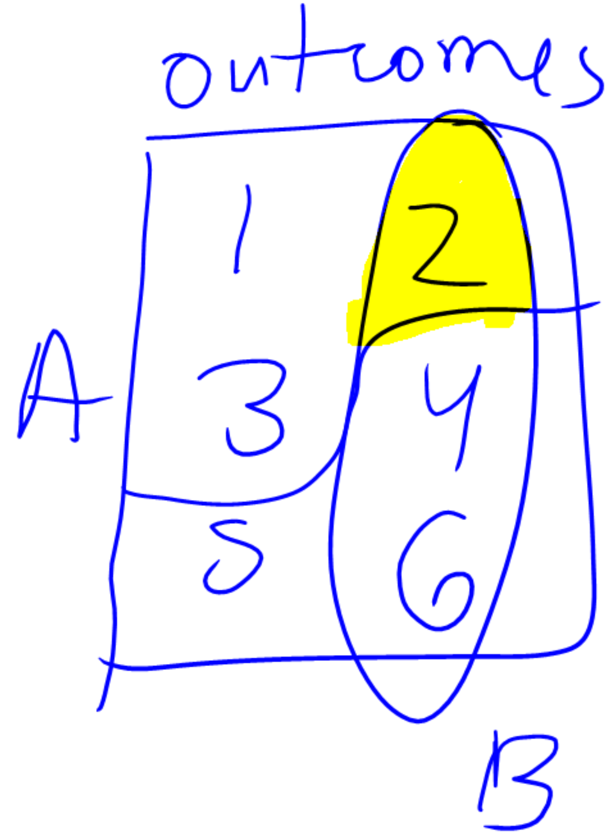
Probability that two events both happen

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that both A and B occur is

$$P(A \text{ and } B) = \frac{\# \text{ of outcomes satisfying both } A \text{ and } B}{\text{total } \# \text{ of outcomes}} = \frac{1}{6}$$

- **Example 2:** I roll a fair six-sided die. What is the probability that the roll is 3 or less and even?

A B



The multiplication rule

- The multiplication rule specifies how to compute the probability of both A and B happening, even if all outcomes are not equally likely.

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

- **Example 2, again:** I roll a fair six-sided die. What is the probability that the roll is $\underbrace{\text{3 or less}}_{A}$ and $\underbrace{\text{even}}_{B}$?

$$P(\text{3 or less AND even}) =$$

$$\underbrace{P(\text{3 or less})}_{1/2} \times \underbrace{P(\text{even given that 3 or less})}_{1/3} = 1/6$$

What if A isn't affected by B? 🤔

- The multiplication rule states that, for any two events A and B,

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

- What if knowing that A happens doesn't tell you anything about the likelihood of B happening?
    - Suppose we flip a fair coin three times.
    - The probability that the $\underbrace{\text{second flip is heads}}$ doesn't depend on the result of the first flip.
- Then, what is $P(A \text{ and } B)$?

B = 2nd flip H

A = 1st flip H

$$P(A \text{ and } B) = P(A) \times \underbrace{P(B \cancel{\text{ given } A})}_{P(B)}$$

irrelevant

$$= \frac{1}{2} \times \frac{1}{2}$$

$$= 1/4$$

same as irrelevant

$P(B \text{ } \cancel{\text{given } A})$

Independent events

- Two events A and B are independent if $P(B \text{ given } A) = P(B)$, or equivalently if $= P(B)$

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

- **Example 3:** Suppose we have a coin that is **biased**, and flips heads with probability 0.7. Each flip is independent of all other flips. We flip it 5 times. What's the probability we see 5 heads in a row?

$$P(1^{st} \text{ flip H AND } 2^{nd} \text{ flip H AND} \ldots)$$

$$= \underbrace{0.7}_{1^{st}} * \underbrace{0.7}_{2^{rd}} * \underbrace{0.7}_{3^{rd}} * \underbrace{0.7}_{4^{th}} * \underbrace{0.7}_{5^{th}} = (0.7)^5$$

Probability that an event *doesn't* happen

- The probability that A **doesn't** happen is $1 - P(A)$.
- For example, if the probability it is sunny tomorrow is 0.85, then the probability it is not sunny tomorrow is 0.15.

$P(\text{talk}) = 1 - P(\text{no talk})$

$= 19/27$

Concept Check ✅ – Answer at <u>cc.dsc10.com</u>

Every time I call my grandma 👵, the probability that she answers her phone is $\frac{1}{3}$. If I call my grandma three times today, what is the chance that I will talk to her at least once?

- A) $\frac{1}{3}$ - should be more
- B) $\frac{2}{3}$
- C) $\frac{1}{2}$
- D) 1
- E) None of the above.

opposite is never talk to grandma

$P(\text{never talk})$

$P(\text{no } 1^{st} \text{ AND no } 2^{nd} \text{ AND no } 3^{rd})$

$P(\text{no } 1^{st}) * P(\text{no } 2^{nd}) * P(\text{no } 3^{rd})$

$2/3 * 2/3 * 2/3 = 8/27$

Probability of either of two events happening

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that either A or B occur is

$$P(A \text{ or } B) = \frac{\# \text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \# \text{ of outcomes}}$$

- **Example 4:** I roll a fair six-sided die. What is the probability that the roll is even or more than 5?

outcomes

1 2
3 4
5 6

$3/6$

even or more than 4

1 2
3 4
5 6

$4/6$

The addition rule

- Suppose that if A happens, then B doesn't, and if B happens, then A doesn't.
  - Such events are called **mutually exclusive** – they have **no overlap**.
- If A and B are any two mutually exclusive events, then

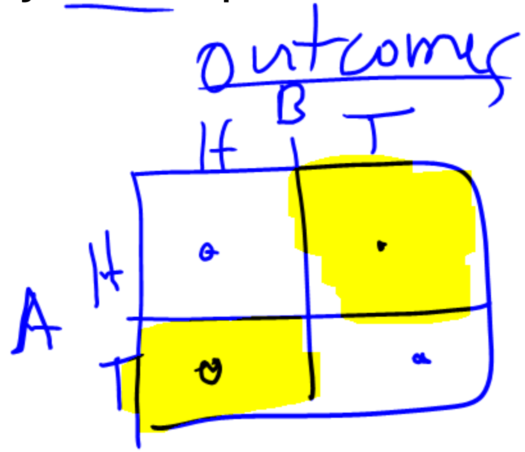
$$\boxed{P(A \text{ or } B) = P(A) + P(B)}$$

- **Example 5:** Suppose I have two biased coins, coin A and coin B. Coin A flips heads with probability 0.6, and coin B flips heads with probability 0.3. I flip both coins once. What's the probability I see two different faces?

$(A = H \text{ and } B = T)$

or $(A = T \text{ and } B = H)$

$= 0.6 * 0.7 + 0.4 * 0.3$



Aside: proof of the addition rule for equally-likely events

You are not required to know how to "prove" anything in this course; you may just find this interesting.

If A and B are events consisting of equally likely outcomes, and furthermore A and B are mutually exclusive (meaning they have no overlap), then

$$\begin{aligned}
P(A \text{ or } B) &= \frac{\#\text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \# \text{ of outcomes}} \\
&= \frac{(\#\text{ of outcomes satisfying } A) + (\#\text{ of outcomes satisfying } B)}{\text{total } \# \text{ of outcomes}} \\
&= \frac{(\#\text{ of outcomes satisfying } A)}{\text{total } \# \text{ of outcomes}} + \frac{(\#\text{ of outcomes satisfying } B)}{\text{total } \# \text{ of outcomes}} \\
&= P(A) + P(B)
\end{aligned}$$

Summary

Summary

- Probability describes the likelihood of an event occurring.
- There are several rules for computing probabilities. We looked at many special cases that involved equally-likely events.
- There are two general rules to be aware of:
    - The **multiplication rule**, which states that for any two events, $P(A \text{ and } B) = P(B \text{ given } A) \cdot P(A)$.
    - The **addition rule**, which states that for any two **mutually exclusive** events, $P(A \text{ or } B) = P(A) + P(B)$.
- **Next time:** simulations.