Lecture 12 – Probability

DSC 10, Fall 2022

Announcements

- Lab 4 is due **Saturday at 11:59PM**.
- Homework 4 is due on **Tuesday 10/25 at 11:59PM**.
- The Midterm Project will be released **today** and is due **Tuesday 11/1.**
    - It takes much longer than a homework, so start now!
    - Partners are optional but recommended, and can be from any lecture section.
    - If you're looking for a partner, email tutor Anna (a2liu@ucsd.edu).
    - You must use the **pair programming** model when working with a partner.

Agenda

We'll cover the basics of probability theory. This is a math lesson; take written notes. ✍🏽

Probability resources

Probability is a tricky subject. If it doesn't click during lecture or on the assignments, take a look at the following resources:

- **Computational and Inferential Thinking, Chapter 9.5**.
- **Theory Meets Data, Chapters 1 and 2**.
- **Khan Academy's unit on Probability**.

Probability theory

Probability theory

- Some things in life *seem* random.
    - e.g. flipping a coin or rolling a die 🎲.
- The **probability** of seeing "heads" when flipping a fair coin is $\frac{1}{2}$.
- One interpretation of probability says that if we flipped a coin infinitely many times, then $\frac{1}{2}$ of the outcomes would be heads.

Terminology

- **Experiment**: A process or action whose result is random.
  - e.g., rolling a die.
  - e.g., flipping a coin twice.
- **Outcome**: The result of an experiment.
  - e.g., the possible outcomes of rolling a six-sided die are 1, 2, 3, 4, 5, and 6.
  - e.g., the possible outcomes of flipping a coin twice are HH, HT, TH, and TT.
- **Event**: A set of outcomes.
  - e.g., the event that the die lands on a even number is the set of outcomes {2, 4, 6}.
  - e.g., the event that the die lands on a 5 is the set of outcomes {5}.
  - e.g., the event that there is at least 1 head in 2 flips is the set of outcomes {HH, HT, TH}.

outcomes

1	2
3	4
5	6

Terminology

- **Probability**: A number between 0 and 1 (equivalently, between 0% and 100%) which describes the likelihood of an event.
    - 0: the event never happens.
    - 1: the event always happens.
- Notation: if A is an event, $P(A)$ is the probability of that event.

Equally-likely outcomes

- If all outcomes in event A are equally likely, then the probability of A is

$$\left[P(A) = \frac{\# \text{ of outcomes satisfying } A}{\text{total } \# \text{ of outcomes}} \right] = 3/8$$

- **Example 1:** Suppose we flip a fair coin 3 times. What is the probability we see exactly 2 heads? A

outcomes

HHH	THH
HTH	TTH
HTT	TTT
HHT	THT

incorrect:

4 outcomes
- 0 H
- 1 H
- 2 H
- 3 H

prob: 1/4

Concept Check ✅ – Answer at cc.dsc10.com

R B G

I have three cards: red, blue, and green. What is the chance that I choose a card at random and it is green, then – **without putting it back** – I choose another card at random and it is red?

- A) $\frac{1}{9}$
- B) $\frac{1}{6}$
- C) $\frac{1}{3}$
- D) $\frac{2}{3}$
- E) None of the above.

$\frac{1}{3}$ AND $\frac{1}{2}$

$\times$

G on 1st R on 2nd

multiplication rule

outcomes

RB BR GR

RG BG GB

1/6

Conditional probabilities

- Two events A and B can both happen. Suppose that we know A has happened, but we don't know if B has.
- If all outcomes are equally likely, then the conditional probability of B given A is:

we know in advance
that A happened

$$P(B \text{ given } A) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\#\text{ of outcomes satisfying } A}$$

- Intuitively, this is similar to the definition of the regular probability of B, $P(B) = \frac{\#\text{ of outcomes satisfying } B}{\text{total }\#\text{ of outcomes}}$, if you restrict the set of possible outcomes to be just those in event A.

P(B given A)

Concept Check ✅ – Answer at cc.dsc10.com

$$\left[P(B \text{ given } A) = \frac{\# \text{ of outcomes satisfying both } A \text{ and } B}{\# \text{ of outcomes satisfying } A}\right] = \frac{1}{3}$$

I roll a six-sided die and don't tell you what the result is, but I tell you that it is 3 or less. What is the probability that the result is even?

- ~~A) $\frac{1}{2}$~~
- B) $\frac{1}{3}$
- C) $\frac{1}{4}$
- D) None of the above.

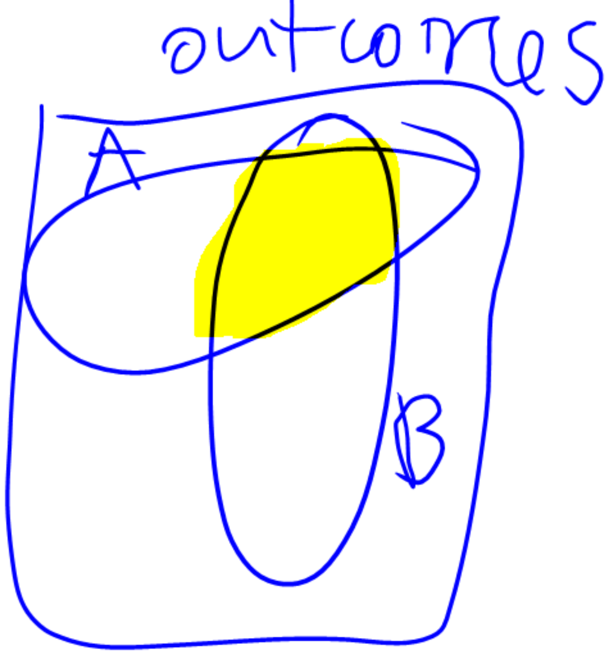
$$P(\underbrace{\text{even}}_{B} \text{ given } \underbrace{\text{3 or less}}_{A})$$

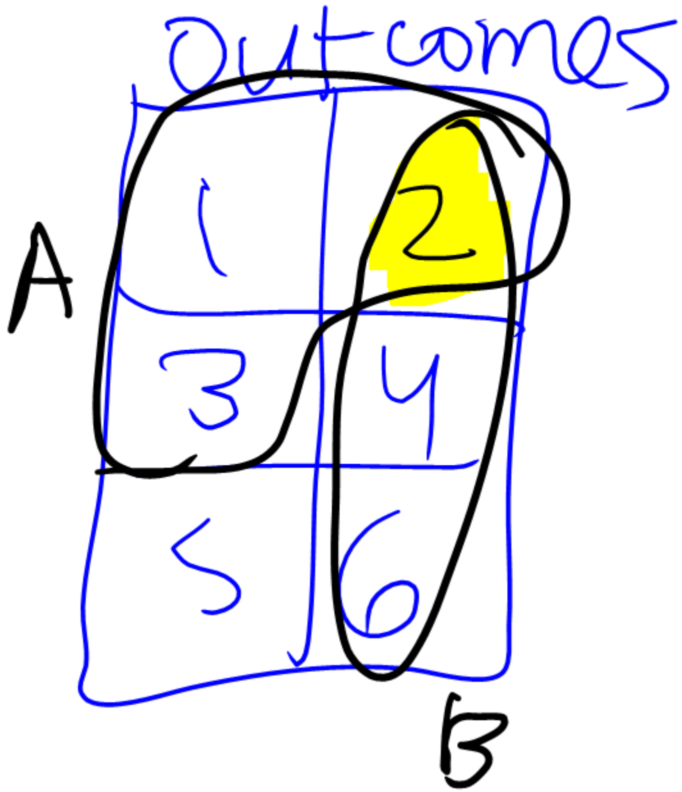
Probability that two events both happen

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that both A and B occur is

$$P(A \text{ and } B) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\text{total } \#\text{ of outcomes}} = 1/6$$

- **Example 2:** I roll a fair six-sided die. What is the probability that the roll is 3 or less and even?





The multiplication rule

- The multiplication rule specifies how to compute the probability of both A and B happening, even if all outcomes are not equally likely.

$$P(A \text{ and } B) = \underbrace{P(A)}_{\text{1st}} \cdot \underbrace{P(B \text{ given } A)}_{\text{2nd}}$$

- **Example 2, again:** I roll a fair six-sided die. What is the probability that the roll is 3 or less and even?

$$\underbrace{\text{is 3 or less}}_{A} \text{ and } \underbrace{\text{even}}_{B}$$

$$P(A \text{ and } B) = P(A) * P(B \text{ given } A)$$

$1/2$ *

$1/3$

$= 1/6$

What if A isn't affected by B? 🤔

- The multiplication rule states that, for any two events A and B,

$$P(A \text{ and } B) = P(A) \cdot P(B \cancel{\text{ given } A})$$

sometimes irrelevant

$P(B \text{ given } A) = P(B)$ when A, B are independent

- What if knowing that A happens doesn't tell you anything about the likelihood of B happening?
  - Suppose we flip a fair coin three times.
  - The probability that the second flip is heads doesn't depend on the result of the first flip.
- Then, what is $P(A \text{ and } B)$?

equivalently

P(B given A) = P(B)

Independent events

- Two events A and B are independent if $P(B \text{ given } A) = P(B)$, or equivalently if

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

- **Example 3:** Suppose we have a coin that is **biased**, and flips heads with probability 0.7. Each flip is independent of all other flips. We flip it 5 times. What's the probability we see 5 heads in a row?

"1st flip H" 5th flip H

$P(A \text{ and } B \text{ and } C \text{ and } D \text{ and } E)$

$= P(A) * P(B) * P(C) * P(D) * P(E)$ ← bc. ind.

$= 0.7 * 0.7 * 0.7 * 0.7 * 0.7$

$= (0.7)^5$

Probability that an event *doesn't* happen

- The probability that A **doesn't** happen is $1 - P(A)$.
- For example, if the probability it is sunny tomorrow is 0.85, then the probability it is not sunny tomorrow is 0.15.

$P(A)$ $1-P(A)$

Concept Check ✅ – Answer at cc.dsc10.com

Every time I call my grandma 👵, the probability that she answers her phone is $\frac{1}{3}$. If I call my grandma three times today, what is the chance that I will talk to her at least once?

- ~~A) $\frac{1}{3}$~~
- B) $\frac{2}{3}$
- C) $\frac{1}{2}$
- ~~D) 1~~
- E) None of the above.

$P(\text{talk at least once}) = 1 - P(\text{never talk})$

$= 1 - \left(\frac{2}{3}\right) * \left(\frac{2}{3}\right) * \left(\frac{2}{3}\right)$

$= \frac{19}{27}$

not talk 1st call

not talk 2nd call

not talk 3rd call

Probability of either of two events happening

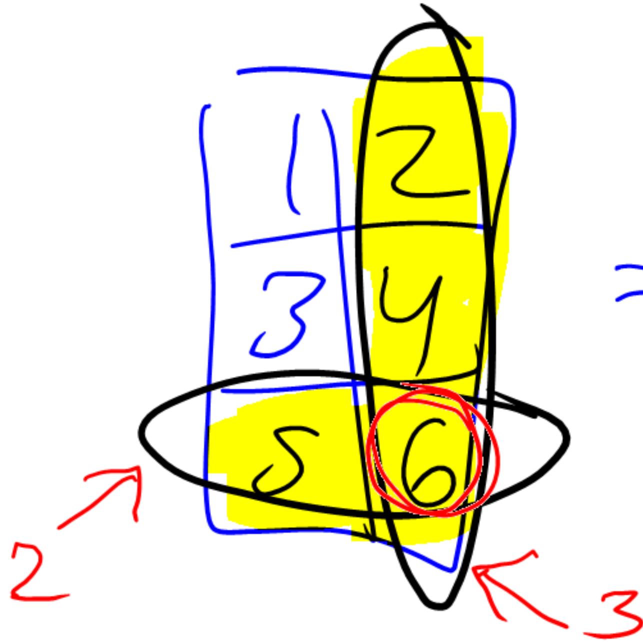
- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that either A or B occur is

$$P(A \text{ or } B) = \frac{\#\text{ of outcomes satisfying either } A \text{ or } B}{\text{total }\#\text{ of outcomes}}$$

- **Example 4:** I roll a fair six-sided die. What is the probability that the roll is even or more than 5?

$= \frac{3}{6}$

$P(\text{even or} \geq 5)$



$= \frac{4}{6}$

The addition rule

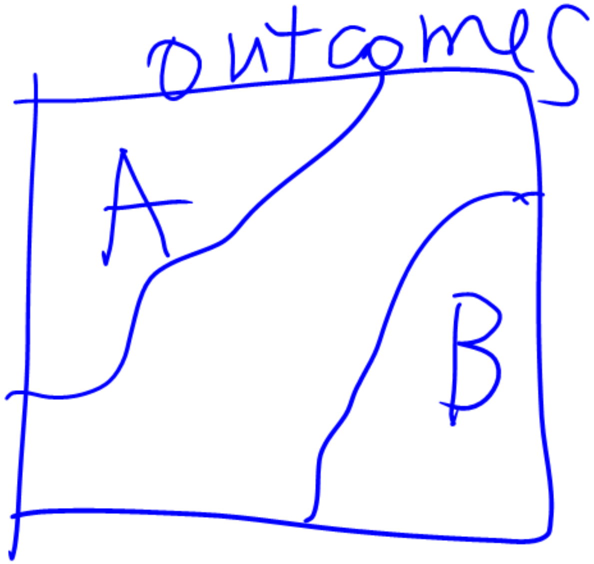
- Suppose that if A happens, then B doesn't, and if B happens, then A doesn't.
  - Such events are called **mutually exclusive** – they have **no overlap**.
- If A and B are any two mutually exclusive events, then

$$P(A \text{ or } B) = P(A) + P(B)$$

- **Example 5:** Suppose I have two biased coins, coin A and coin B. Coin A flips heads with probability 0.6, and coin B flips heads with probability 0.3. I flip both coins once. What's the probability I see two different faces?

$0.42 + 0.12$

0.54



outcomes

A	B
H	H
H	T
T	H
T	T

H AND T

$0.6 * 0.7$

T AND H

$0.4 * 0.3$

Aside: proof of the addition rule for equally-likely events

You are not required to know how to "prove" anything in this course; you may just find this interesting.

If A and B are events consisting of equally likely outcomes, and furthermore A and B are mutually exclusive (meaning they have no overlap), then

$$\begin{aligned}
P(A \text{ or } B) &= \frac{\# \text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \# \text{ of outcomes}} \\
&= \frac{(\# \text{ of outcomes satisfying } A) + (\# \text{ of outcomes satisfying } B)}{\text{total } \# \text{ of outcomes}} \\
&= \frac{(\# \text{ of outcomes satisfying } A)}{\text{total } \# \text{ of outcomes}} + \frac{(\# \text{ of outcomes satisfying } B)}{\text{total } \# \text{ of outcomes}} \\
&= P(A) + P(B)
\end{aligned}$$

Summary

Summary

- Probability describes the likelihood of an event occurring.
- There are several rules for computing probabilities. We looked at many special cases that involved equally-likely events.
- There are two general rules to be aware of:
    - The **multiplication rule**, which states that for any two events, $P(A \text{ and } B) = P(B \text{ given } A) \cdot P(A)$.
    - The **addition rule**, which states that for any two **mutually exclusive** events, $P(A \text{ or } B) = P(A) + P(B)$.
- **Next time:** simulations.