Lecture 12 – Probability

DSC 10, Fall 2022

Announcements

- Lab 4 is due **Saturday at 11:59PM**.
- Homework 4 is due on **Tuesday 10/25 at 11:59PM**.
- The Midterm Project will be released **today** and is due **Tuesday 11/1.**
    - It takes much longer than a homework, so start now!
    - Partners are optional but recommended, and can be from any lecture section.
    - If you're looking for a partner, email tutor Anna (a2liu@ucsd.edu).
    - You must use the **pair programming** model when working with a partner.

Agenda

We'll cover the basics of probability theory. This is a math lesson; take written notes. ✍🏽

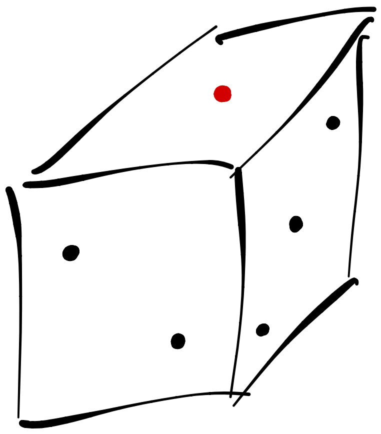
Probability resources

Probability is a tricky subject. If it doesn't click during lecture or on the assignments, take a look at the following resources:

- **Computational and Inferential Thinking, Chapter 9.5**.
- **Theory Meets Data, Chapters 1 and 2**.
- **Khan Academy's unit on Probability**.

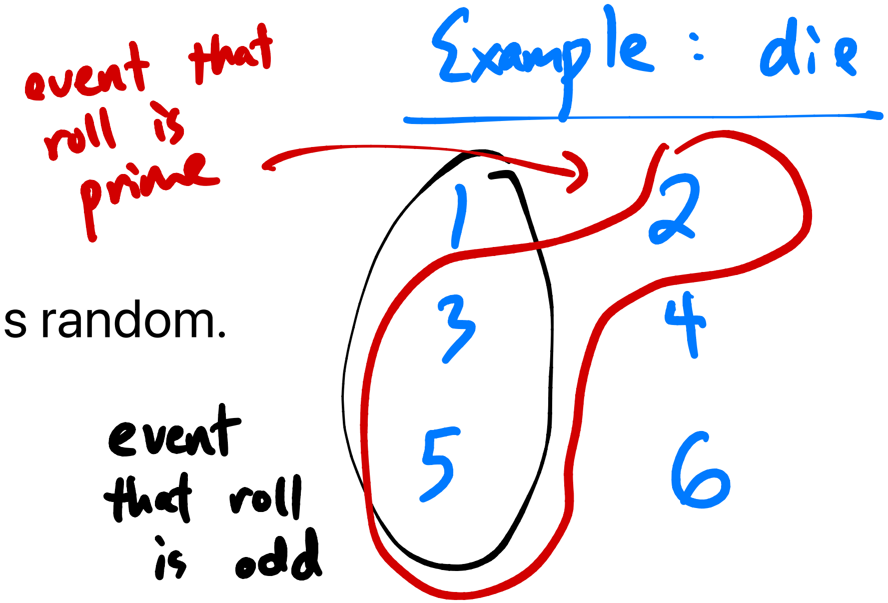
link broken; see Resources tab of course website

Probability theory



Probability theory

- Some things in life *seem* random.
  - e.g. flipping a coin or rolling a die 🎲.
- The **probability** of seeing "heads" when flipping a fair coin is $\frac{1}{2}$.
- One interpretation of probability says that if we flipped a coin infinitely many times, then $\frac{1}{2}$ of the outcomes would be heads.



Terminology

- **Experiment**: A process or action whose result is random.
    - e.g., rolling a die.
    - e.g., flipping a coin twice.
- **Outcome**: The result of an experiment.
    - e.g., the possible outcomes of rolling a six-sided die are 1, 2, 3, 4, 5, and 6.
    - e.g., the possible outcomes of flipping a coin twice are HH, HT, TH, and TT.
- **Event**: A set of outcomes.
    - e.g., the event that the die lands on a even number is the set of outcomes {2, 4, 6}.
    - e.g., the event that the die lands on a 5 is the set of outcomes {5}.
    - e.g., the event that there is at least 1 head in 2 flips is the set of outcomes {HH, HT, TH}.

Terminology

- **Probability**: A number between 0 and 1 (equivalently, between 0% and 100%) which describes the likelihood of an event.
    - 0: the event never happens.
    - 1: the event always happens.
- Notation: if A is an event, $P(A)$ is the probability of that event.

$P(A)$ $P(3 \text{ or less})$

Equally-likely outcomes

- If all outcomes in event A are equally likely, then the probability of A is

$$P(A) = \frac{\#\text{ of outcomes satisfying } A}{\text{total }\#\text{ of outcomes}}$$

- **Example 1:** Suppose we flip a fair coin 3 times. What is the probability we see exactly 2 heads?

Outcomes:

HHH THH
HHT THT
HTH TTH
HTT TTT

$$P(\text{exactly 2 heads}) = \frac{3}{8}$$

Concept Check ✅ – Answer at **cc.dsc10.com**

I have three cards: red, blue, and green. What is the chance that I choose a card at random and it is green, then – **without putting it back** – I choose another card at random and it is red?

- A) $\frac{1}{9}$
- B) $\frac{1}{6}$
- C) $\frac{1}{3}$
- D) $\frac{2}{3}$
- E) None of the above.

Outcomes:

RG RB
GR GB
BR BG

P(Green then Red) = $\frac{1}{6}$

Conditional probabilities

- Two events A and B can both happen. Suppose that we know A has happened, but we don't know if B has.
- If all outcomes are equally likely, then the conditional probability of B given A is:

$$P(B \text{ given } A) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\#\text{ of outcomes satisfying } A}$$

- Intuitively, this is similar to the definition of the regular probability of B, $P(B) = \frac{\#\text{ of outcomes satisfying } B}{\text{total } \#\text{ of outcomes}}$, if you restrict the set of possible outcomes to be just those in event A.

P(exactly 2 heads given at least 1 head)

$$= \frac{3}{7}$$

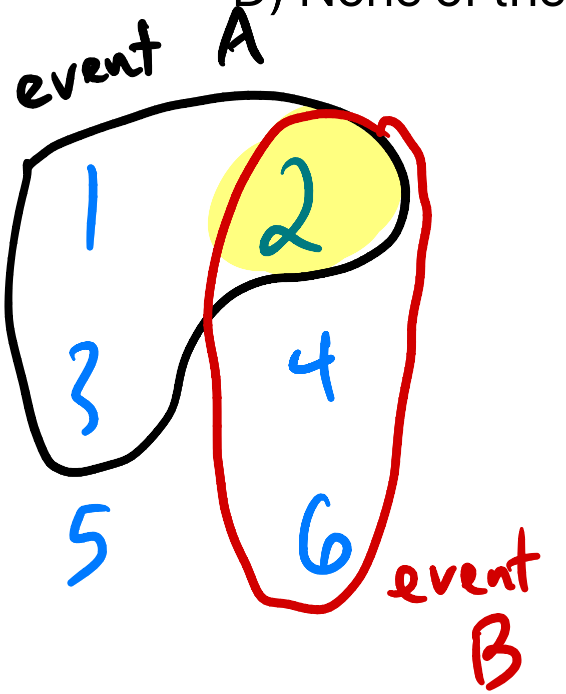
HHH	THH
HHT	THT
HTH	TTH
HTT	TTT

Concept Check ✅ – Answer at cc.dsc10.com

$$P(B \text{ given } A) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\#\text{ of outcomes satisfying } A}$$

I roll a six-sided die and don't tell you what the result is, but I tell you that it is 3 or less. What is the probability that the result is even?

- A) $\frac{1}{2}$
- B) $\frac{1}{3}$
- C) $\frac{1}{4}$
- D) None of the above.



$$P(\text{even given 3 or less})$$

$$= \frac{\#\text{ outcomes 3 or less AND even}}{\#\text{ outcomes 3 or less}}$$

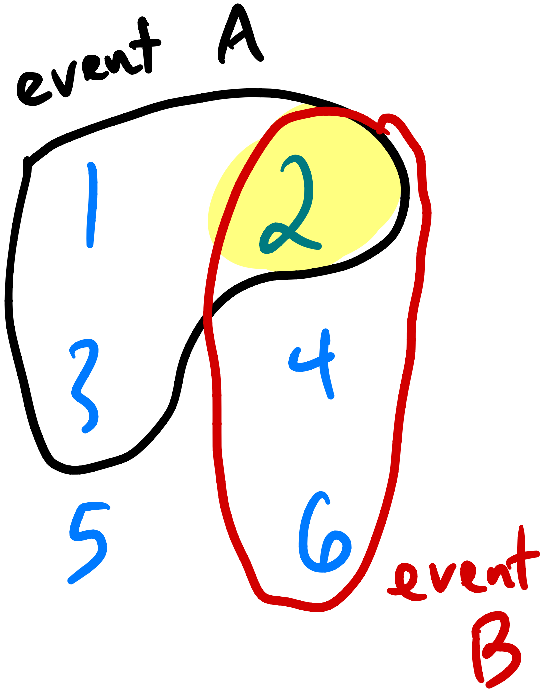
$$= \frac{1}{3}$$

Probability that two events both happen

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that both A and B occur is

$$P(A \text{ and } B) = \frac{\# \text{ of outcomes satisfying both } A \text{ and } B}{\text{total } \# \text{ of outcomes}}$$

- **Example 2:** I roll a fair six-sided die. What is the probability that the roll is 3 or less and even?



$$P(3 \text{ or less and even}) = \frac{1}{6}$$

The multiplication rule

- The multiplication rule specifies how to compute the probability of both A and B happening, even if all outcomes are not equally likely.

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

general multiplication rule

- **Example 2, again:** I roll a fair six-sided die. What is the probability that the roll is 3 or less and even?

$$P(3 \text{ or less and even}) = P(3 \text{ or less}) \cdot P(\text{even given } 3 \text{ or less})$$

$$= \frac{1}{2} \cdot \frac{1}{3}$$

from earlier

$$= \frac{1}{6}$$

What if A isn't affected by B? 🤔

- The multiplication rule states that, for any two events A and B,

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

- What if knowing that A happens doesn't tell you anything about the likelihood of B happening?
  - Suppose we flip a fair coin three times.
  - The probability that the second flip is heads doesn't depend on the result of the first flip.
- Then, what is $P(A \text{ and } B)$?

independent:

$P(B \text{ given } A) = P(B)$

not true in general!

Independent events

- Two events A and B are independent if $P(B \text{ given } A) = P(B)$, or equivalently if

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

- **Example 3:** Suppose we have a coin that is **biased**, and flips heads with probability 0.7. Each flip is independent of all other flips. We flip it 5 times. What's the probability we see 5 heads in a row?

$$P(\text{all 5 heads}) = P(1^{st} \text{ heads and } 2^{nd} \text{ heads and} \dots \text{ and } 5^{th} \text{ heads})$$

$$= P(1^{st} \text{ heads}) \cdot P(2^{nd} \text{ heads}) \cdot \dots \cdot P(5^{th} \text{ heads})$$

$$= 0.7 \cdot 0.7 \cdot 0.7 \cdot 0.7 \cdot 0.7$$

$$= \boxed{0.7^5}$$

Probability that an event *doesn't* happen

- The probability that A **doesn't** happen is $1 - P(A)$.
- For example, if the probability it is sunny tomorrow is 0.85, then the probability it is not sunny tomorrow is 0.15.

Concept Check ✅ – Answer at cc.dsc10.com

Every time I call my grandma 👵, the probability that she answers her phone is $\frac{1}{3}$. If I call my grandma three times today, what is the chance that I will talk to her at least once?

- A) $\frac{1}{3}$
- B) $\frac{2}{3}$
- C) $\frac{1}{2}$
- ~~D) 1~~
- E) None of the above.

$$P(\text{answers at least once}) = 1 - P(\text{never answers})$$

assume calls independent!

$$= 1 - P(\text{doesn't answer call 1}) \cdot P(\text{no call 2}) \cdot P(\text{no call 3})$$

$$= 1 - \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3}$$

$$= 1 - \underbrace{\frac{8}{27}} = \boxed{\frac{19}{27}}$$

Probability of either of two events happening

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that either A or B occur is

$$P(A \text{ or } B) = \frac{\#\text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \#\text{ of outcomes}}$$

- **Example 4:** I roll a fair six-sided die. What is the probability that the roll is even or more than 5?

1 2 ✓

3 4 ✓

5 ✓ 6 ✓ event B

event A

$$P(A \text{ or } B) = \frac{4}{6} = \frac{2}{3}$$

The addition rule

- Suppose that if A happens, then B doesn't, and if B happens, then A doesn't.
  - Such events are called **mutually exclusive** – they have **no overlap**.
- If A and B are any two mutually exclusive events, then

$$P(A \text{ or } B) = P(A) + P(B)$$

- **Example 5:** Suppose I have two biased coins, coin A and coin B. Coin A flips heads with probability 0.6, and coin B flips heads with probability 0.3. I flip both coins once. What's the probability I see two different faces?

$$P(\text{different faces}) = P\underbrace{(A \text{ heads}, B \text{ tails})} + P\underbrace{(A \text{ tails}, B \text{ heads})}$$

$$= 0.6 \cdot (1-0.3) + (1-0.6) \cdot 0.3$$

$$= 0.42 + 0.12 = \boxed{0.54}$$

Aside: proof of the addition rule for equally-likely events

You are not required to know how to "prove" anything in this course; you may just find this interesting.
If A and B are events consisting of equally likely outcomes, and furthermore A and B are mutually exclusive (meaning they have no overlap), then

$$
\begin{aligned}
P(A \text{ or } B) &= \frac{\#\text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \#\text{ of outcomes}} \\
&= \frac{(\#\text{ of outcomes satisfying } A) + (\#\text{ of outcomes satisfying } B)}{\text{total } \#\text{ of outcomes}} \\
&= \frac{(\#\text{ of outcomes satisfying } A)}{\text{total } \#\text{ of outcomes}} + \frac{(\#\text{ of outcomes satisfying } B)}{\text{total } \#\text{ of outcomes}} \\
&= P(A) + P(B)
\end{aligned}
$$

Summary

Summary

- Probability describes the likelihood of an event occurring.
- There are several rules for computing probabilities. We looked at many special cases that involved equally-likely events.
- There are two general rules to be aware of:
    - The **multiplication rule**, which states that for any two events, $P(A \text{ and } B) = P(B \text{ given } A) \cdot P(A)$.
    - The **addition rule**, which states that for any two **mutually exclusive** events, $P(A \text{ or } B) = P(A) + P(B)$.
- **Next time:** simulations.