Lecture 11 – Probability

DSC 10, Spring 2023

Announcements

- Lab 3 is due on **Saturday 4/29 at 11:59PM**.
- Homework 3 is due on **Tuesday 5/2 at 11:59PM**.
- The Midterm Project is due on **Tuesday, 5/9 at 11:59PM**.
    - You can work with a partner from any lecture section, but if you do, you must follow these **project partner guidelines**. In particular, you must both contribute to all parts of the project and not split up the problems.
    - We will hold a mixer **today from 1:30 to 2:15PM in the Center Hall courtyard** to help you find a partner if you don't have one. See **this Ed post** for questions.
    - Start early!
- **Tomorrow from 12-1:30PM in CSE 1202**, a few other DSC professors and I will be participating in a "**Hot Ones**"-style event hosted by DS3. Come say hi!

Last time: `for`-loops

- Almost every `for`-loop in DSC 10 will use the **accumulator pattern**.
    - This means we initialize a variable, and repeatedly add on to it within a loop.
    - The variable could be an integer, an array, or even a string (as in Homework 3, Question 4: Triton Tweets).
    - Analogy: Start with a blank piece of paper and write something on it each time you run an experiment.
- Do **not** use `for`-loops to perform mathematical operations on every element of an array or Series.
    - Instead, use DataFrame manipulations and built-in array or Series methods.
- Helpful video 🎥: **For Loops (and when not to use them) in DSC 10**.
- More examples to come in Friday's lecture.

Agenda

We'll cover the basics of probability theory. This is a math lesson; take written notes ✍🏽.

Probability resources

Probability is a tricky subject. If it doesn't click during lecture or on the assignments, take a look at the following resources:

- **Computational and Inferential Thinking, Chapter 9.5**.
- **Theory Meets Data, Chapters 1 and 2**.
- **Khan Academy's unit on Probability**.

We're also going to review probability again in next Wednesday's lecture (the lecture before the midterm).

Probability theory

- Some things in life *seem* random.
    - e.g., flipping a coin or rolling a die 🎲.
- The **probability** of seeing "heads" when flipping a fair coin is $\frac{1}{2}$.
- One interpretation of probability says that if we flipped a coin infinitely many times, then $\frac{1}{2}$ of the outcomes would be heads.

Example : Die

1 2

3 4

5 6

event that roll is prime

event that roll is even

Terminology

- **Experiment**: A process or action whose result is random.
    - e.g., rolling a die.
    - e.g., flipping a coin twice.
- **Outcome**: The result of an experiment.
    - e.g., the possible outcomes of rolling a six-sided die are 1, 2, 3, 4, 5, and 6.
    - e.g., the possible outcomes of flipping a coin twice are HH, HT, TH, and TT.
- **Event**: A set of outcomes.
    - e.g., the event that the die lands on a even number is the set of outcomes {2, 4, 6}.
    - e.g., the event that the die lands on a 5 is the set of outcomes {5}.
    - e.g., the event that there is at least 1 head in 2 flips is the set of outcomes {HH, HT, TH}.

Terminology

- **Probability**: A number between 0 and 1 (equivalently, between 0% and 100%) that describes the likelihood of an event.
    - 0: The event never happens.
    - 1: The event always happens.
- Notation: If A is an event, $P(A)$ is the probability of that event.

P(roll is prime)   P(roll a 2)

WRONG!

0H 1H $\in$ not equally likely!

2H 3H

Equally-likely outcomes

- If all outcomes in event A are equally likely, then the probability of A is

$$P(A) = \frac{\#\text{ of outcomes satisfying } A}{\text{total } \#\text{ of outcomes}}$$

- **Example 1**: Suppose we flip a fair coin 3 times. What is the probability we see exactly 2 heads?

8 possible outcomes $(2 \times 2 \times 2)$

HHH THH

HHT THT

HTH TTH

HTT TTT

$$\boxed{\text{probability} = \frac{3}{8}}$$

Concept Check ✅ – Answer at cc.dsc10.com

I have three cards: red, blue, and green. What is the chance that I choose a card at random and it is green, then – **without putting it back** – I choose another card at random and it is red?

- A) $\frac{1}{9}$
- B) $\frac{1}{6}$
- C) $\frac{1}{3}$
- D) $\frac{2}{3}$
- E) None of the above.

Outcomes : 6

RG
RB
GR
GB
BR
BG

$\boxed{\text{probability} = \frac{1}{6}}$

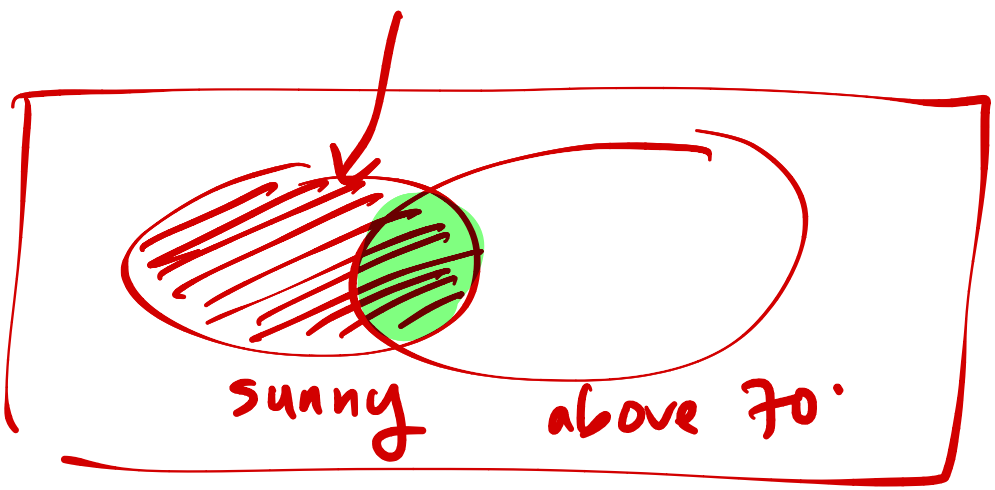
Conditional probabilities

- Two events A and B can both happen. Suppose that we know A has happened, but we don't know if B has.
- If all outcomes are equally likely, then the conditional probability of B given A is:

$$P(B \text{ given } A) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\#\text{ of outcomes satisfying } A}$$

given (know it happens)

- Intuitively, this is similar to the definition of the regular probability of B, $P(B) = \frac{\#\text{ of outcomes satisfying } B}{\text{total }\#\text{ of outcomes}}$, if you restrict the set of possible outcomes to be just those in event A.



Concept Check ✅ – Answer at **cc.dsc10.com**

$$P(B \text{ given } A) = \frac{\# \text{ of outcomes satisfying both } A \text{ and } B}{\# \text{ of outcomes satisfying } A}$$

I roll a six-sided die and don't tell you what the result is, but I tell you that it is 3 or less. What is the probability that the result is even?

- A) $\frac{1}{2}$
- B) $\frac{1}{3}$
- C) $\frac{1}{4}$
- D) None of the above.

$$P(\text{even given 3 or less}) = \frac{1}{3}$$

1 2
3 4
5 6

3 or less

even

Probability that two events both happen

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that both A and B occur is

$$P(A \text{ and } B) = \frac{\# \text{ of outcomes satisfying both } A \text{ and } B}{\text{total } \# \text{ of outcomes}}$$

- **Example 2**: I roll a fair six-sided die. What is the probability that the roll is 3 or less **and** even?

$$P(3 \text{ or less AND even}) = \frac{\# \text{ outcomes 3 or less AND even}}{\text{total } \#}$$

1 2

3 4

5 6

3 or less

even

$$= \boxed{\frac{1}{6}}$$

The multiplication rule

- The multiplication rule specifies how to compute the probability of both A and B happening, even if all outcomes are not equally likely.

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

- **Example 2, again**: I roll a fair six-sided die. What is the probability that the roll is 3 or less and even?

$$P(3 \text{ or less AND even})$$

$$= P(3 \text{ or less}) \cdot P\left(\begin{array}{c}\text{even} \\ \text{given} \\ 3 \text{ or less}\end{array}\right)$$

$$= \frac{1}{2} \cdot \frac{1}{3}$$

from 2 slides ago

$$= \boxed{\frac{1}{6}}$$

1 2
3 4
5 6

3 or less

even

What if A isn't affected by B? 🤔

- The multiplication rule states that, for any two events A and B,

$$P(A \text{ and } B) = P(A) \cdot \underbrace{P(B \text{ given } A)}$$

- What if knowing that A happens doesn't tell you anything about the likelihood of B happening?
  - Suppose we flip a fair coin three times.
  - The probability that the second flip is heads doesn't depend on the result of the first flip.
- Then, what is $P(A \text{ and } B)$?

when A and B have nothing to do with one another,

$$P(B \text{ given } A) = P(B)$$

AND → multiply

Independent events

$P(2^{nd}\ H)$ = probability that 2^{nd} flip is heads

- Two events A and B are independent if $P(B \text{ given } A) = P(B)$, or equivalently if

$$\underbrace{P(A \text{ and } B) = P(A) \cdot P(B)}$$

→ special case!

- **Example 3**: Suppose we have a coin that is **biased**, and flips heads with probability 0.7. Each flip is independent of all other flips. We flip it 5 times. What's the probability we see 5 heads in a row?

$$P(\text{all 5 heads}) = P(1^{st}\ H \text{ AND } 2^{nd}\ H \text{ AND } \cdots \text{ AND } 5^{th}\ H)$$

$$= P(1^{st}\ H) \cdot P(2^{nd}\ H) \cdot \ldots \cdot P(5^{th}\ H)$$

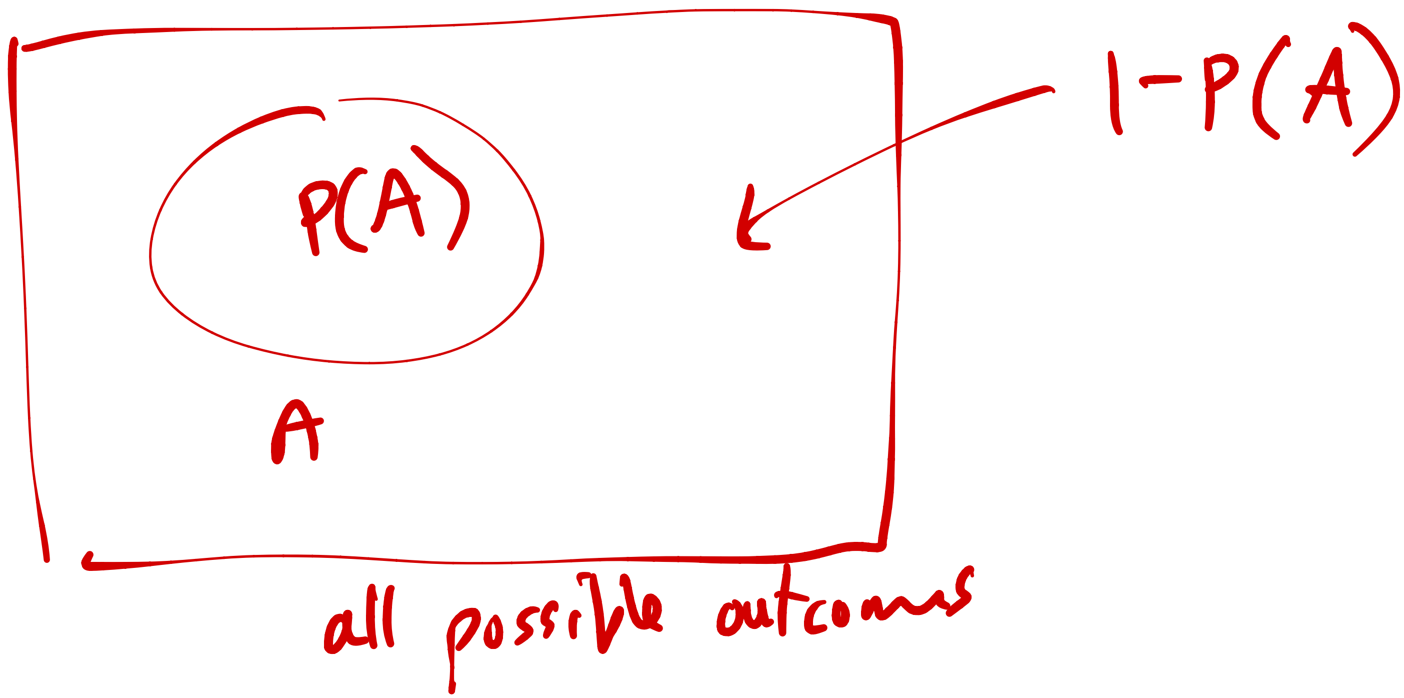
$$= 0.7 \cdot 0.7 \cdot 0.7 \cdot 0.7 \cdot 0.7$$

$$= \boxed{0.7^5}$$

"complement rule"

Probability that an event *doesn't* happen

- The probability that A **doesn't** happen is $1 - P(A)$.
- For example, if the probability it is sunny tomorrow is 0.85, then the probability it is not sunny tomorrow is 0.15.



Concept Check ✅ – Answer at cc.dsc10.com

Every time I call my grandma 👵, the probability that she answers her phone is $\frac{1}{3}$, independently for each call. If I call my grandma three times today, what is the chance that I will talk to her at least once?

- A) $\frac{1}{3}$
- B) $\frac{2}{3}$
- C) $\frac{1}{2}$
- ~~D) 1~~
- E) None of the above.

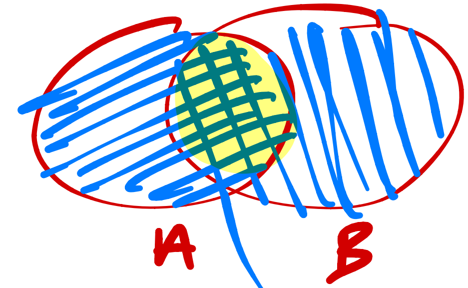
$P(\text{at least once}) = 1 - P(\text{never})$

$= 1 - P(\text{no 1st AND no 2nd AND no 3rd})$

$= 1 - P(\text{no 1st}) \cdot P(\text{no 2nd}) \cdot P(\text{no 3rd})$

$= 1 - \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = 1 - \frac{8}{27} = \boxed{\frac{19}{27}}$

Probability of either of two events happening

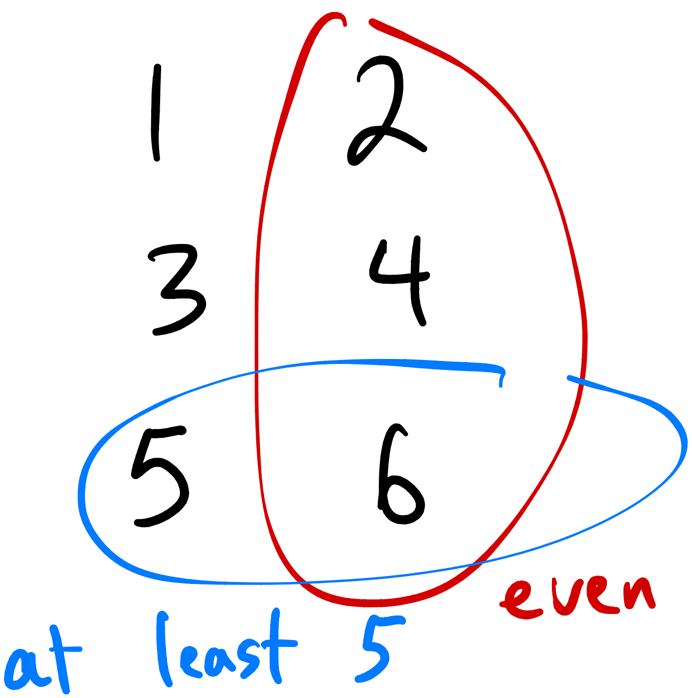


double counted!

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that either A or B occur is

$$P(A \text{ or } B) = \frac{\#\text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \#\text{ of outcomes}}$$

- **Example 4**: I roll a fair six-sided die. What is the probability that the roll is even or at least 5?



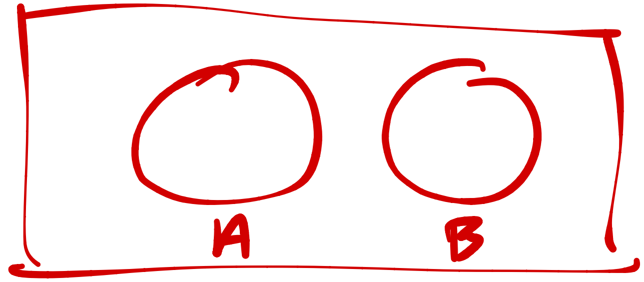
$P(\text{at least } 5) = \frac{2}{6}$ ← not the sum!

$P(\text{even}) = \frac{3}{6}$

$$P(\text{even OR at least } 5) = \frac{4}{6} = \frac{2}{3}$$

OR → add

The addition rule



- Suppose that if A happens, then B doesn't, and if B happens, then A doesn't.
    - Such events are called **mutually exclusive** – they have **no overlap**.
- If A and B are any two mutually exclusive events, then

$$P(A \text{ or } B) = P(A) + P(B)$$

- **Example 5**: Suppose I have two biased coins, coin A and coin B. Coin A flips heads with probability 0.6, and coin B flips heads with probability 0.3. I flip both coins once. What's the probability I see two different faces?

$$P(A=\text{heads AND } B=\text{tails OR } A=\text{tails AND } B=\text{heads})$$
$$= P(A=\text{heads AND } B=\text{tails}) + P(A=\text{tails AND } B=\text{heads})$$
$$= P(A=\text{heads}) \cdot P(B=\text{tails}) + P(A=\text{tails}) \cdot P(B=\text{heads})$$
$$= 0.6 \cdot (1-0.3) + (1-0.6) \cdot 0.3 = 0.42 + 0.12 = \boxed{0.54}$$

Aside: Proof of the addition rule for equally-likely events

You are not required to know how to "prove" anything in this course; you may just find this interesting.

If A and B are events consisting of equally likely outcomes, and furthermore A and B are mutually exclusive (meaning they have no overlap), then

$$\begin{aligned}
P(A \text{ or } B) &= \frac{\#\text{ of outcomes satisfying either } A \text{ or } B}{\text{total }\#\text{ of outcomes}} \\
&= \frac{(\#\text{ of outcomes satisfying } A) + (\#\text{ of outcomes satisfying } B)}{\text{total }\#\text{ of outcomes}} \\
&= \frac{(\#\text{ of outcomes satisfying } A)}{\text{total }\#\text{ of outcomes}} + \frac{(\#\text{ of outcomes satisfying } B)}{\text{total }\#\text{ of outcomes}} \\
&= P(A) + P(B)
\end{aligned}$$

Summary, next time

- Probability describes the likelihood of an event occurring.
- There are several rules for computing probabilities. We looked at many special cases that involved equally-likely events.
- There are two general rules to be aware of:
  - The **multiplication rule**, which states that for any two events, $P(A \text{ and } B) = P(B \text{ given } A) \cdot P(A)$.
  - The **addition rule**, which states that for any two **mutually exclusive** events, $P(A \text{ or } B) = P(A) + P(B)$.
- **Next time:** Simulations.