Lecture 11 – Probability

DSC 10, Winter 2023

Announcements

- Lab 3 is due **tomorrow at 11:59PM**.
- Homework 3 is due **Tuesday 2/7 at 11:59PM**.
- The Midterm Project (Restaurants 🍔🍟) is released and due **Tuesday 2/14 at 11:59PM**.
    - Partners must follow **these partner guidelines**. In particular, you must both contribute to all parts of the project and not split up the problems.
    - Still looking for a partner? Use **this thread on EdStem** to connect with classmates.

Last time: `for`-loops

- Almost every `for`-loop in DSC 10 will use the **accumulator pattern**.
    - This means we initialize a variable, and repeatedly add on to it within a loop.
    - The variable could be an integer, an array, or even a string (as in Homework 3, Question 4: Wordle 🟨⬛🟨🟩⬛).
- Do **not** use `for`-loops to perform mathematical operations on every element of an array or Series.
    - Instead use DataFrame manipulations and built-in array or Series methods.
- Helpful video 🎥: **[For Loops (and when not to use them) in DSC 10]()**.

Agenda

We'll cover the basics of probability theory. This is a math lesson; take written notes. ✍🏽

Probability resources

Probability is a tricky subject. If it doesn't click during lecture or on the assignments, take a look at the following resources:

- **Computational and Inferential Thinking, Chapter 9.5**.
- **Theory Meets Data, Chapters 1 and 2**.
- **Khan Academy's unit on Probability**.

Probability theory

- Some things in life *seem* random.
  - e.g. flipping a coin or rolling a die 🎲.
- The **probability** of seeing "heads" when flipping a fair coin is $\frac{1}{2}$.
- One interpretation of probability says that if we flipped a coin infinitely many times, then $\frac{1}{2}$ of the outcomes would be heads.

Terminology

- **Experiment**: A process or action whose result is random.
  - e.g., rolling a die.
  - e.g., flipping a coin twice.
- **Outcome**: The result of an experiment.
  - e.g., the possible outcomes of rolling a six-sided die are 1, 2, 3, 4, 5, and 6.
  - e.g., the possible outcomes of flipping a coin twice are HH, HT, TH, and TT.
- **Event**: A set of outcomes.
  - e.g., the event that the die lands on a even number is the set of outcomes {2, 4, 6}.
  - e.g., the event that the die lands on a 5 is the set of outcomes {5}.

- e.g., the event that there is at least 1 head in 2 flips is the set of outcomes {HH, HT, TH}.

Terminology

- **Probability**: A number between 0 and 1 (equivalently, between 0% and 100%) which describes the likelihood of an event.
    - 0: the event never happens.
    - 1: the event always happens.
- Notation: if A is an event, $P(A)$ is the probability of that event.

Equally-likely outcomes

- If all outcomes in event A are equally likely, then the probability of A is

$$P(A) = \frac{\#\text{ of outcomes satisfying A}}{\text{total }\#\text{ of outcomes}} = \frac{\#\text{ good outcomes}}{\text{total }\#\text{ outcomes}}$$

- **Example 1:** Suppose we flip a fair coin 3 times. What is the probability we see exactly 2 heads?

8 outcomes $= 2 * 2 * 2$

HHH HHT
HTH HTT
THH THT
TTH TTT

$\frac{3}{8}$

incorrect outcomes

0H 1H
2H 3H

$= \frac{1}{4}$

Concept Check ✅ – Answer at **cc.dsc10.com**

I have three cards: red, blue, and green. What is the chance that I choose a card at random and it is green, then – **without putting it back** – I choose another card at random and it is red?

- A) $\frac{1}{9}$
- B) $\frac{1}{6}$
- C) $\frac{1}{3}$
- D) $\frac{2}{3}$
- E) None of the above.

outcomes

GR GB
RB RG
BG BR

$= \frac{1}{6}$

another approach

R
B
G

$= \frac{1}{3}$

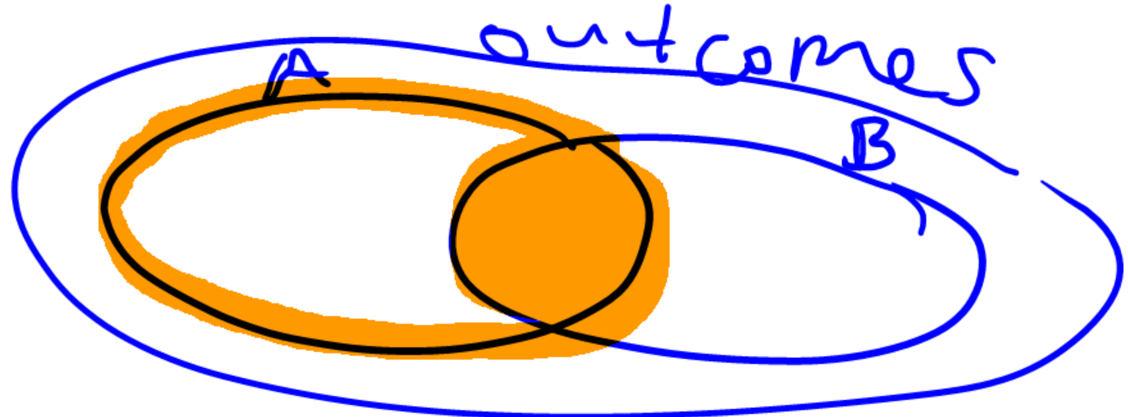
1st card outcomes

R
B

$= \frac{1}{2}$

2nd card outcomes

$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$



Conditional probabilities

- Two events A and B can both happen. Suppose that we know A has happened, but we don't know if B has.
- If all outcomes are equally likely, then the conditional probability of B given A is:

$$P(\text{B given A}) = \frac{\text{\# of outcomes satisfying both A and B}}{\text{\# of outcomes satisfying A}}$$

given A means "knowing that A happened"

- Intuitively, this is similar to the definition of the regular probability of B, $P(B) = \frac{\text{\# of outcomes satisfying B}}{\text{total \# of outcomes}}$, if you restrict the set of possible outcomes to be just those in event A.

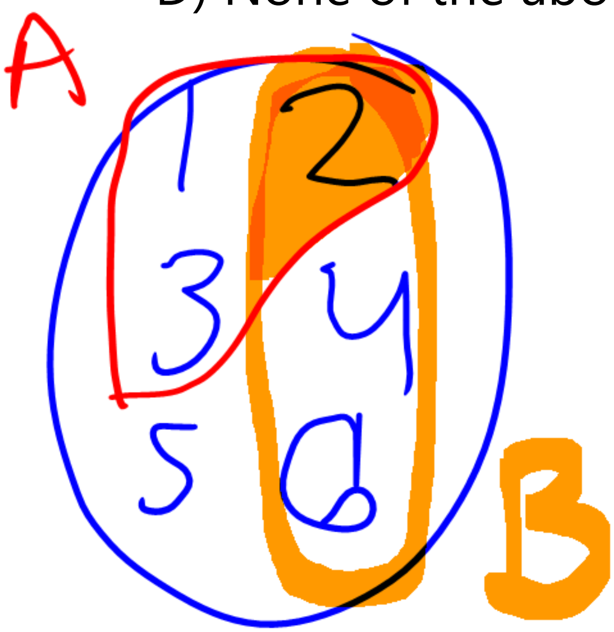
Concept Check ✅ – Answer at **cc.dsc10.com**

$$P(\text{B given A}) = \frac{\#\text{ of outcomes satisfying both A and B}}{\#\text{ of outcomes satisfying A}}$$

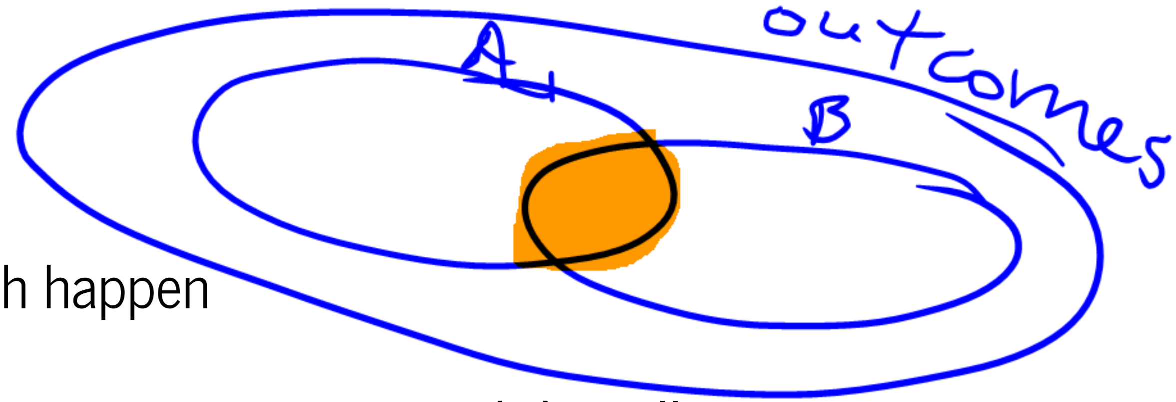
I roll a six-sided die and don't tell you what the result is, but I tell you that it is 3 or less. What is the probability that the result is even?

- A) $\frac{1}{2}$
- B) $\frac{1}{3}$
- C) $\frac{1}{4}$
- D) None of the above.

$$P(\underbrace{\text{even}}_{B} \text{ given } \underbrace{\le 3}_{A})$$



$$= \frac{1}{3}$$

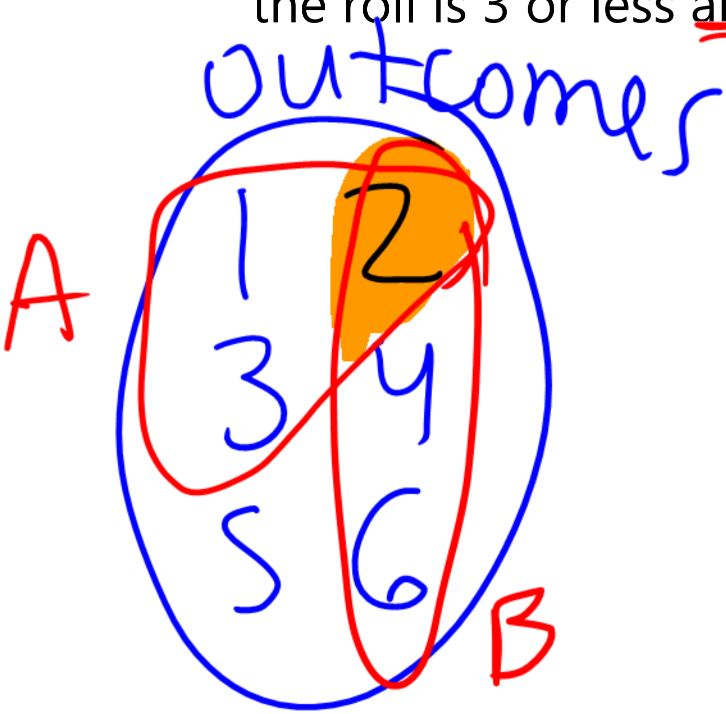


Probability that two events both happen

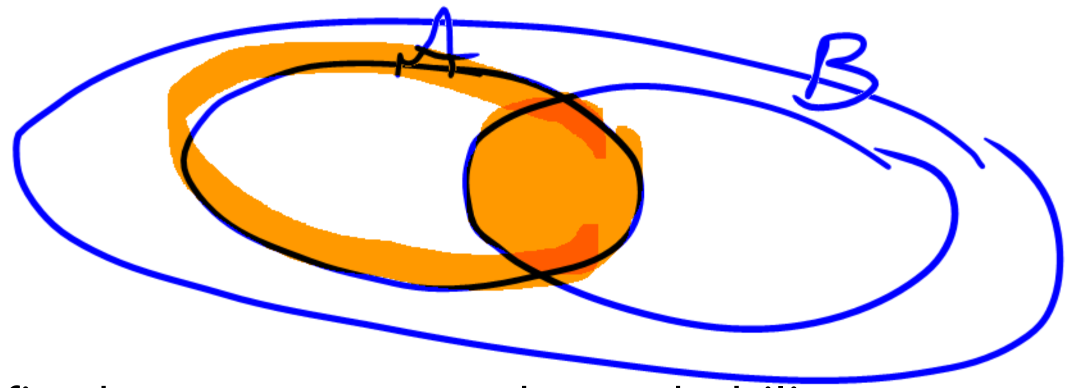
- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that both A and B occur is

$$P(\text{A and B}) = \frac{\#\text{ of outcomes satisfying both A and B}}{\text{total }\#\text{ of outcomes}}$$

- **Example 2:** I roll a fair six-sided die. What is the probability that the roll is 3 or less and even?



$\frac{1}{6}$

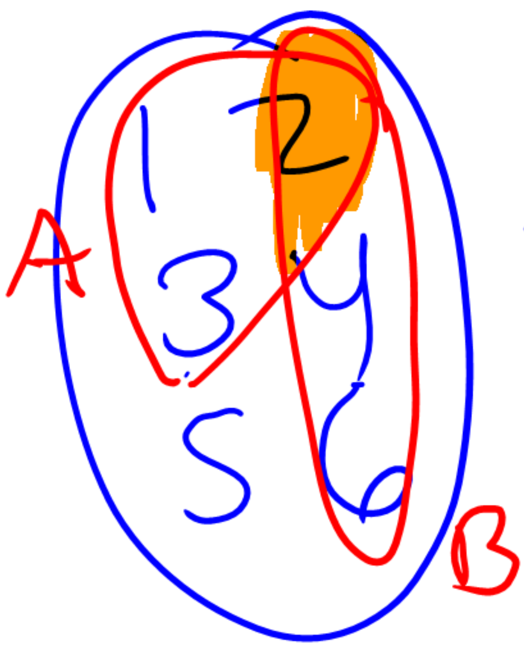


The multiplication rule

- The multiplication rule specifies how to compute the probability of both A and B happening, even if all outcomes are not equally likely.

$$\boxed{P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)}$$

- **Example 2, again:** I roll a fair six-sided die. What is the probability that the roll is $\underbrace{\text{3 or less}}_{A}$ and $\underbrace{\text{even}}_{B}$?



$$P(A \text{ and } B) = P(A) * P(B \text{ given } A)$$

$$= \frac{3}{6} * \frac{1}{3} = \frac{1}{6}$$

What if A isn't affected by B? 🤔

- The multiplication rule states that, for any two events A and B,

$$P(A \text{ and } B) = P(A) \cdot \boxed{P(B \text{ given } A)} \quad \text{Cond. prob.}$$

- What if knowing that A happens doesn't tell you anything about the likelihood of B happening?
    - Suppose we flip a fair coin three times.
    - The probability that the second flip is heads doesn't depend on the result of the first flip.
- Then, what is $P(A \text{ and } B)$?

RGB cards

$$P(G_{1st} \text{ and } R_{2nd}) = P(G_{1st}) * P(R_{2nd} \text{ given } G_{1st})$$

$$= \frac{1}{3} * \frac{1}{2} = \frac{1}{6}$$

Independent events

- Two events A and B are independent if P(B given A) = P(B), or equivalently if

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

- **Example 3:** Suppose we have a coin that is **biased**, and flips heads with probability 0.7. Each flip is independent of all other flips. We flip it 5 times. What's the probability we see 5 heads in a row?

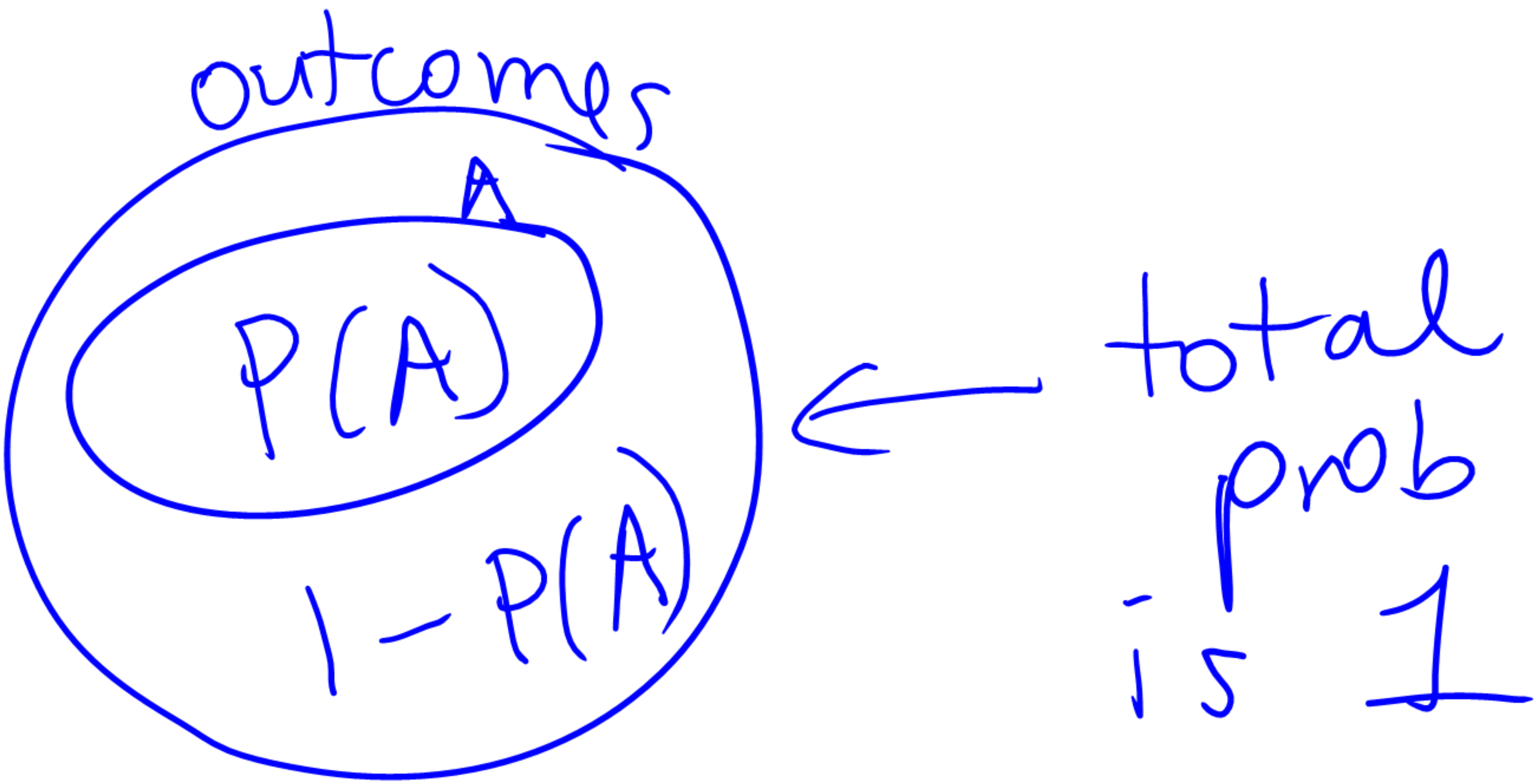
$$P(1^{st} H \text{ AND } 2^{nd} H \text{ AND } \ldots \text{ AND } 5^{th} H)$$

$$= \underbrace{P(1^{st} H)}_{0.7} * \underbrace{P(2^{nd} H)}_{0.7} * \ldots * P(5^{th} H)$$

$$= 0.7 * 0.7 \times \ldots * 0.7 = \boxed{(0.7)^5}$$

Probability that an event *doesn't* happen

- The probability that A **doesn't** happen is $1 - P(A)$.
- For example, if the probability it is sunny tomorrow is 0.85, then the probability it is not sunny tomorrow is 0.15.



$P(\text{talk at least once}) = 1 - P(NNN) = 1 - \left(\frac{2}{3}\right)^3 = \boxed{\frac{19}{27}}$

Y = yes

N = no

Concept Check ✅ – Answer at **cc.dsc10.com**

Every time I call my grandma 👵, the probability that she answers her phone is $\frac{1}{3}$, independently for each call. If I call my grandma three times today, what is the chance that I will talk to her at least once?

$\boxed{\frac{19}{27}}$

- A) $\frac{1}{3}$
- B) $\frac{2}{3}$
- C) $\frac{1}{2}$
- ~~D) 1~~
- E) None of the above.

outcomes

YYY YNN

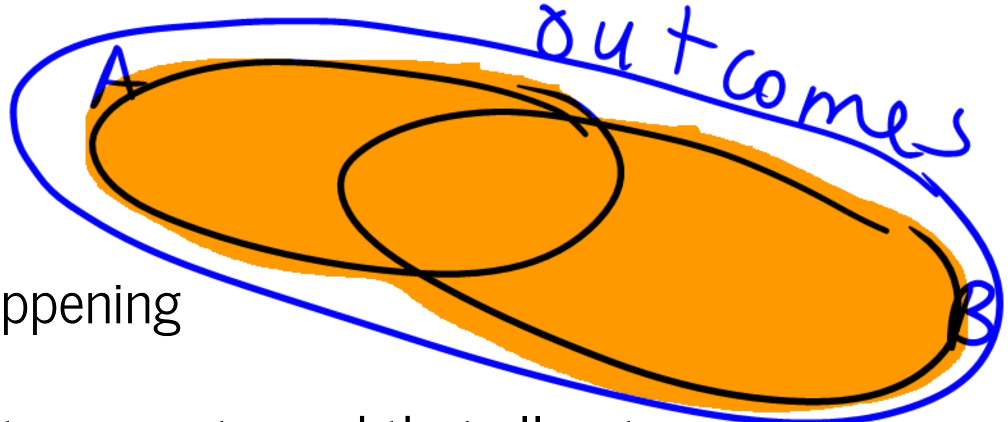
YYN YNY

$\boxed{\text{NYY}}$ NNY

NYN NNN

$P(N \text{ on } 1^{st} \text{ AND } Y \text{ on } 2^{nd} \text{ AND } Y \text{ on } 3^{rd})$

$\frac{2}{3} * \frac{1}{3} * \frac{1}{3} = 2/27$



Probability of either of two events happening

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that either A or B occur is

$$P(A \text{ or } B) = \frac{\text{\# of outcomes satisfying either A or B}}{\text{total \# of outcomes}}$$

- **Example 4:** I roll a fair six-sided die. What is the probability that the roll is even or at least 5?

$= \frac{4}{6}$

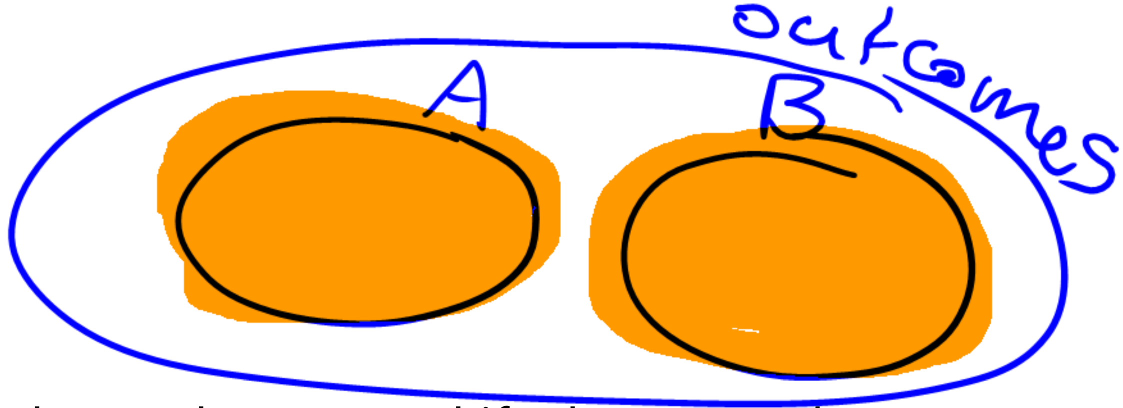
can't just do $P(A)+P(B)$ because A and B overlap

$P(A) = 3/6$

$P(B) = 2/6$

$> 5/6$

issue: double count 6



The addition rule

- Suppose that if A happens, then B doesn't, and if B happens, then A doesn't.
    - Such events are called **mutually exclusive** – they have **no overlap**.
- If A and B are any two mutually exclusive events, then

$$\boxed{P(A \text{ or } B) = P(A) + P(B)}$$

- **Example 5:** Suppose I have two biased coins, coin A and coin B. Coin A flips heads with probability 0.6, and coin B flips heads with probability 0.3. I flip both coins once. What's the probability I see two different faces?

OR

$\underbrace{\text{H on A}}_{0.6}$ AND $\underbrace{\text{T on B}}_{0.7}$ + $\underbrace{\text{T on A}}_{0.4}$ AND $\underbrace{\text{H on B}}_{0.3}$

0.6 * 0.7 + 0.4 * 0.3

Aside: proof of the addition rule for equally-likely events

You are not required to know how to "prove" anything in this course; you may just find this interesting.

If A and B are events consisting of equally likely outcomes, and furthermore A and B are mutually exclusive (meaning they have no overlap), then

$$
\begin{aligned}
P(\text{A or B}) &= \frac{\text{\# of outcomes satisfying either A or B}}{\text{total \# of outcomes}} \\
&= \frac{(\text{\# of outcomes satisfying A}) + (\text{\# of outcomes satisfying B})}{\text{total \# of outcomes}} \\
&= \frac{(\text{\# of outcomes satisfying A})}{\text{total \# of outcomes}} + \frac{(\text{\# of outcomes satisfying B})}{\text{total \# of outcomes}} \\
&= P(A) + P(B)
\end{aligned}
$$

Summary, next time

- Probability describes the likelihood of an event occurring.
- There are several rules for computing probabilities. We looked at many special cases that involved equally-likely events.
- There are two general rules to be aware of:

AND

  - The **multiplication rule**, which states that for any two events, $P(A \text{ and } B) = P(B \text{ given } A) \cdot P(A)$.

OR

  - The **addition rule**, which states that for any two **mutually exclusive** events, $P(A \text{ or } B) = P(A) + P(B)$.
- **Next time:** simulations.