



# Lecture 11 – Probability

DSC 10, Winter 2023


## Announcements

- Lab 3 is due **tomorrow at 11:59PM.**
- Homework 3 is due **Tuesday 2/7 at 11:59PM.**
- The Midterm Project (Restaurants 🍔 🍷) is released and due **Tuesday 2/14 at 11:59PM.**
  - Partners must follow **these partner guidelines**. In particular, you must both contribute to all parts of the project and not split up the problems.
  - Still looking for a partner? Use **this thread on EdStem** to connect with classmates.

Last time: for-loops

- Almost every for-loop in DSC 10 will use the **accumulator pattern**.
  - This means we initialize a variable, and repeatedly add on to it within a loop.
  - The variable could be an integer, an array, or even a string (as in Homework 3, Question 4: Wordle ).
- Do **not** use for-loops to perform mathematical operations on every element of an array or Series.
  - Instead use DataFrame manipulations and built-in array or Series methods.
- Helpful video : **For Loops (and when not to use them) in DSC 10.**

## Agenda


We'll cover the basics of probability theory. This is a math lesson; take written notes. 

## Probability resources

Probability is a tricky subject. If it doesn't click during lecture or on the assignments, take a look at the following resources:

- **Computational and Inferential Thinking, Chapter 9.5.**
- **Theory Meets Data, Chapters 1 and 2.**
- **Khan Academy's unit on Probability.**

## Probability theory

- Some things in life *seem* random.
  - e.g. flipping a coin or rolling a die .
- The **probability** of seeing "heads" when flipping a fair coin is  $\frac{1}{2}$ .
- One interpretation of probability says that if we flipped a coin infinitely many times, then  $\frac{1}{2}$  of the outcomes would be heads.

## Terminology

- **Experiment:** A process or action whose result is random.
  - e.g., rolling a die.
  - e.g., flipping a coin twice.
- **Outcome:** The result of an experiment.
  - e.g., the possible outcomes of rolling a six-sided die are 1, 2, 3, 4, 5, and 6.
  - e.g., the possible outcomes of flipping a coin twice are HH, HT, TH, and TT.
- **Event:** A set of outcomes.
  - e.g., the event that the die lands on an even number is the set of outcomes {2, 4, 6}.
  - e.g., the event that the die lands on a 5 is the set of outcomes {5}.



- e.g., the event that there is at least 1 head in 2 flips is the set of outcomes {HH, HT, TH}.



## Terminology

- **Probability:** A number between 0 and 1 (equivalently, between 0% and 100%) which describes the likelihood of an event.
  - 0: the event never happens.
  - 1: the event always happens.
- Notation: if  $A$  is an event,  $P(A)$  is the probability of that event.

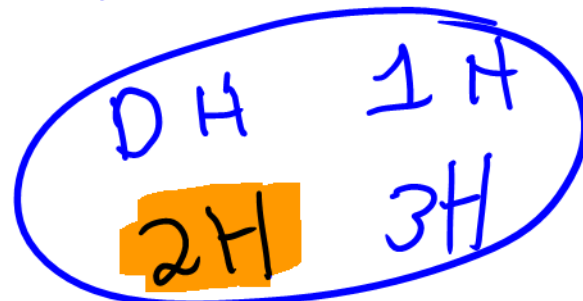
Equally-likely outcomes

- If all outcomes in event A are equally likely, then the probability of A is

$$P(A) = \frac{\text{\# of outcomes satisfying A}}{\text{total \# of outcomes}}$$

- **Example 1:** Suppose we flip a fair coin ~~3~~<sup>2</sup> times. What is the probability we see exactly 2 heads?

outcomes



answer =  $\frac{3}{8}$

3 times

outcomes



answer:  $\frac{1}{4}$



Concept Check  – Answer at [cc.dsc10.com](http://cc.dsc10.com)

I have three cards: red, blue, and green. What is the chance that I choose a card at random and it is **green**, then – **without putting it back** – I choose another card at random and it is **red**?

- A)  $\frac{1}{9}$
- B)  $\frac{1}{6}$
- C)  $\frac{1}{3}$
- D)  $\frac{2}{3}$
- E) None of the above.

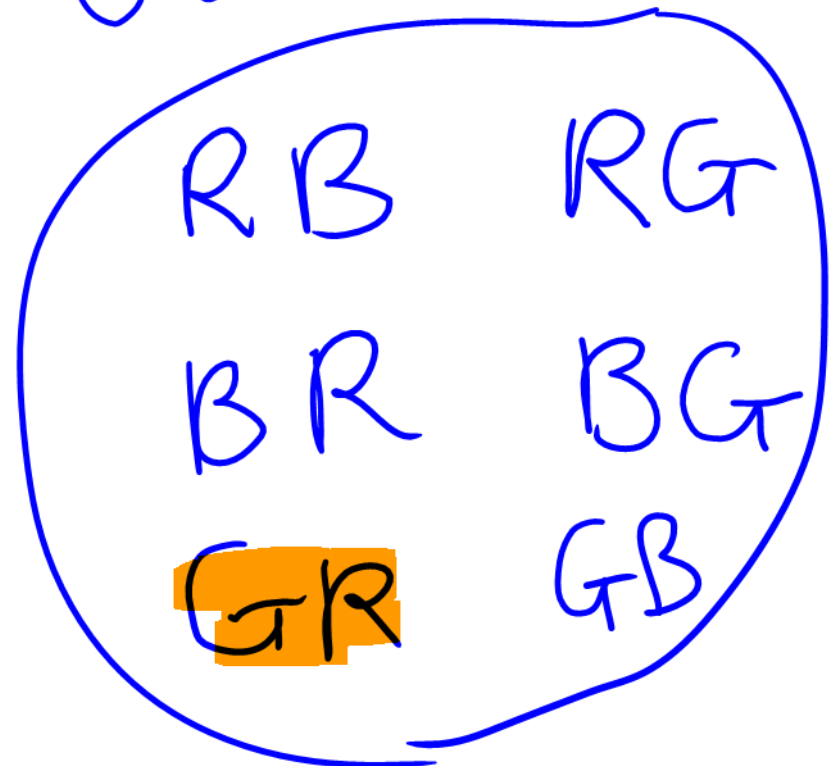
Outcomes



Outcomes



Outcomes





## Conditional probabilities

- Two events A and B can both happen. Suppose that we know A has happened, but we don't know if B has.
- If all outcomes are equally likely, then the conditional probability of B given A is:

$$P(B \text{ given } A) = \frac{\text{\# of outcomes satisfying both A and B}}{\text{\# of outcomes satisfying A}}$$

given A:  
we know  
A  
already  
happened

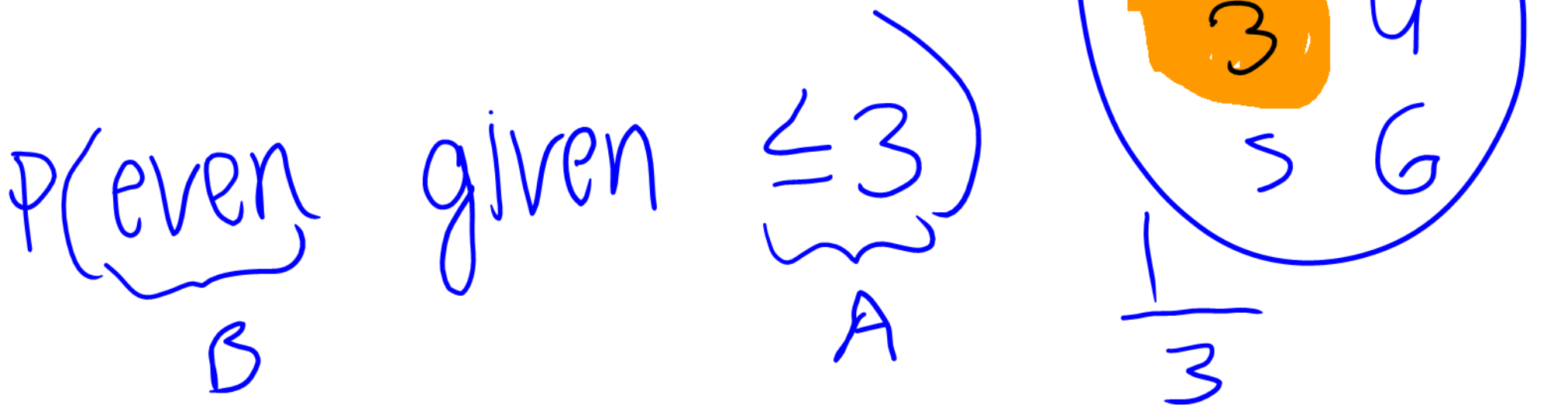
- Intuitively, this is similar to the definition of the regular probability of B,  $P(B) = \frac{\text{\# of outcomes satisfying B}}{\text{total \# of outcomes}}$ , if you restrict the set of possible outcomes to be just those in event A.

Concept Check  – Answer at [cc.dsc10.com](http://cc.dsc10.com)

$$P(B \text{ given } A) = \frac{\text{\# of outcomes satisfying both A and B}}{\text{\# of outcomes satisfying A}}$$

I roll a six-sided die and don't tell you what the result is, but I tell you that it is 3 or less. What is the probability that the result is even?

- A)  $\frac{1}{2}$
- B)  $\frac{1}{3}$
- C)  $\frac{1}{4}$
- D) None of the above.



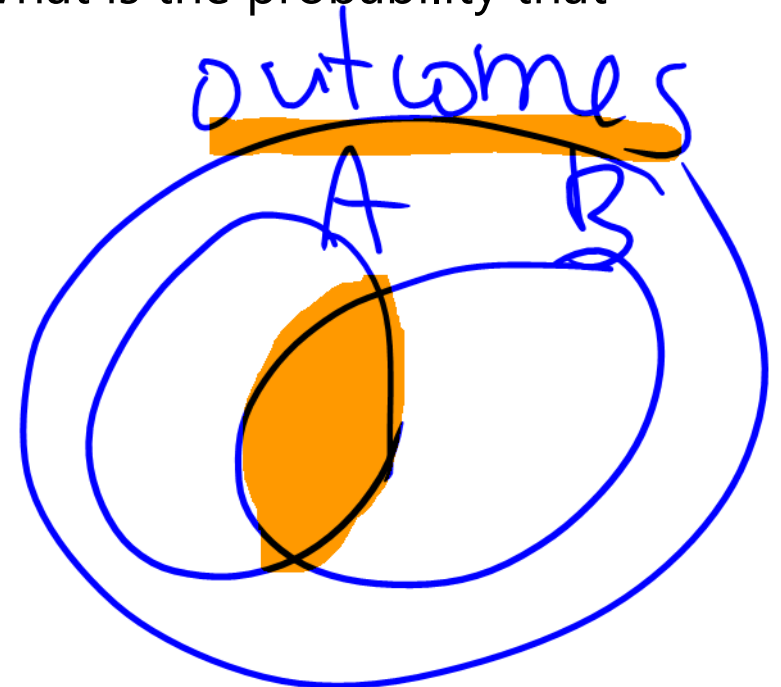
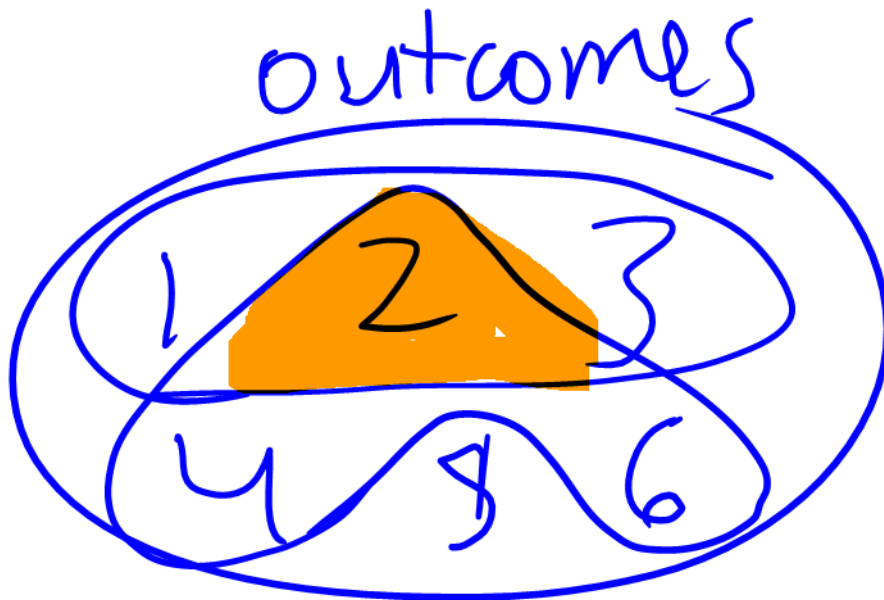
## Probability that two events both happen

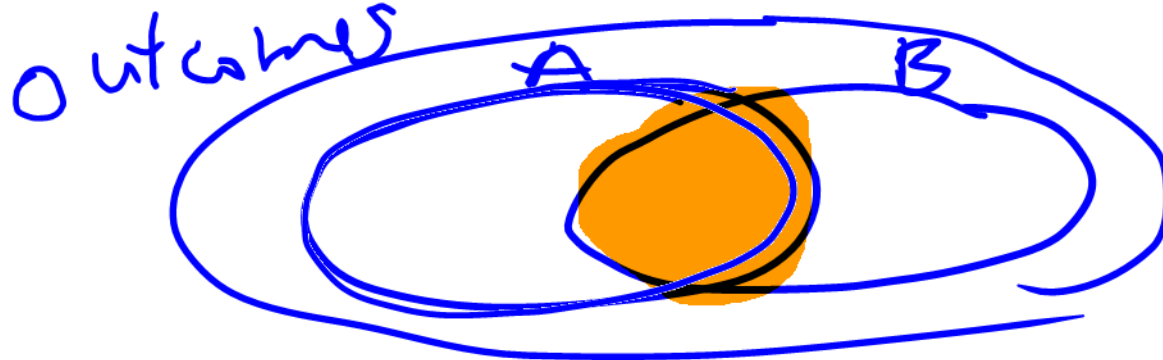
- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that both A and B occur is

$$P(A \text{ and } B) = \frac{\text{\# of outcomes satisfying both A and B}}{\text{total \# of outcomes}}$$

- **Example 2:** I roll a fair six-sided die. What is the probability that the roll is 3 or less and even?

$\frac{1}{6}$





The multiplication rule

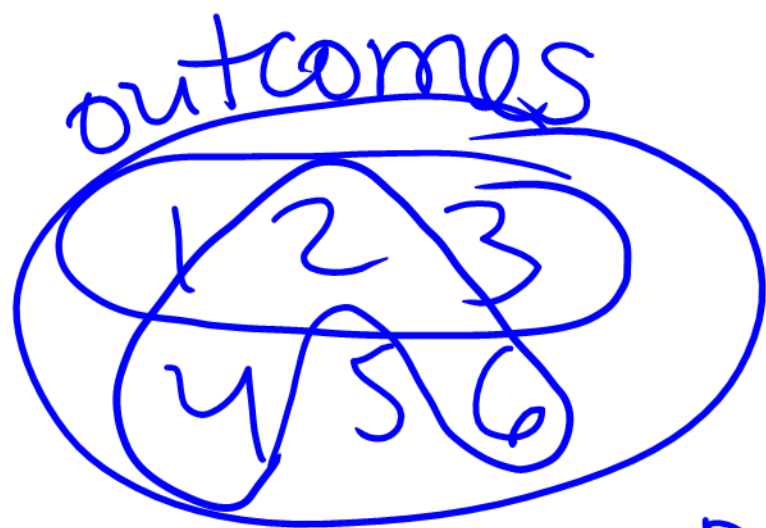
- The multiplication rule specifies how to compute the probability of both A and B happening, even if all outcomes are not equally likely.

$$\frac{3}{6} * \frac{1}{3} = \frac{1}{6}$$

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

RGB Cards  
 $P(G \text{ and } R)$

- Example 2, again:** I roll a fair six-sided die. What is the probability that the roll is 3 or less and even?



$\underbrace{\hspace{2cm}}_A \quad \underbrace{\hspace{2cm}}_B$

$$P(A) = \frac{3}{6}$$

$$P(B \text{ given } A) = \frac{1}{3}$$

$$= P(G) * P(R \text{ given } G) = \frac{1}{3} * \frac{1}{2}$$

What if  $A$  isn't affected by  $B$ ? 😞

- The multiplication rule states that, for any two events  $A$  and  $B$ ,

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

- What if knowing that  $A$  happens doesn't tell you anything about the likelihood of  $B$  happening?

- Suppose we flip a fair coin three times.
- The probability that the second flip is heads doesn't depend on the result of the first flip.

- Then, what is  $P(A \text{ and } B)$ ?  $= P(A) * P(B)$

may just be the same as  $P(B)$

$$P(\text{H on 1st flip and H on 2nd})$$
$$= P(\text{H on 1st flip}) * P(\text{H on 2nd given H on 1st})$$



## Independent events

- Two events A and B are independent if  $P(B \text{ given } A) = P(B)$ , or equivalently if

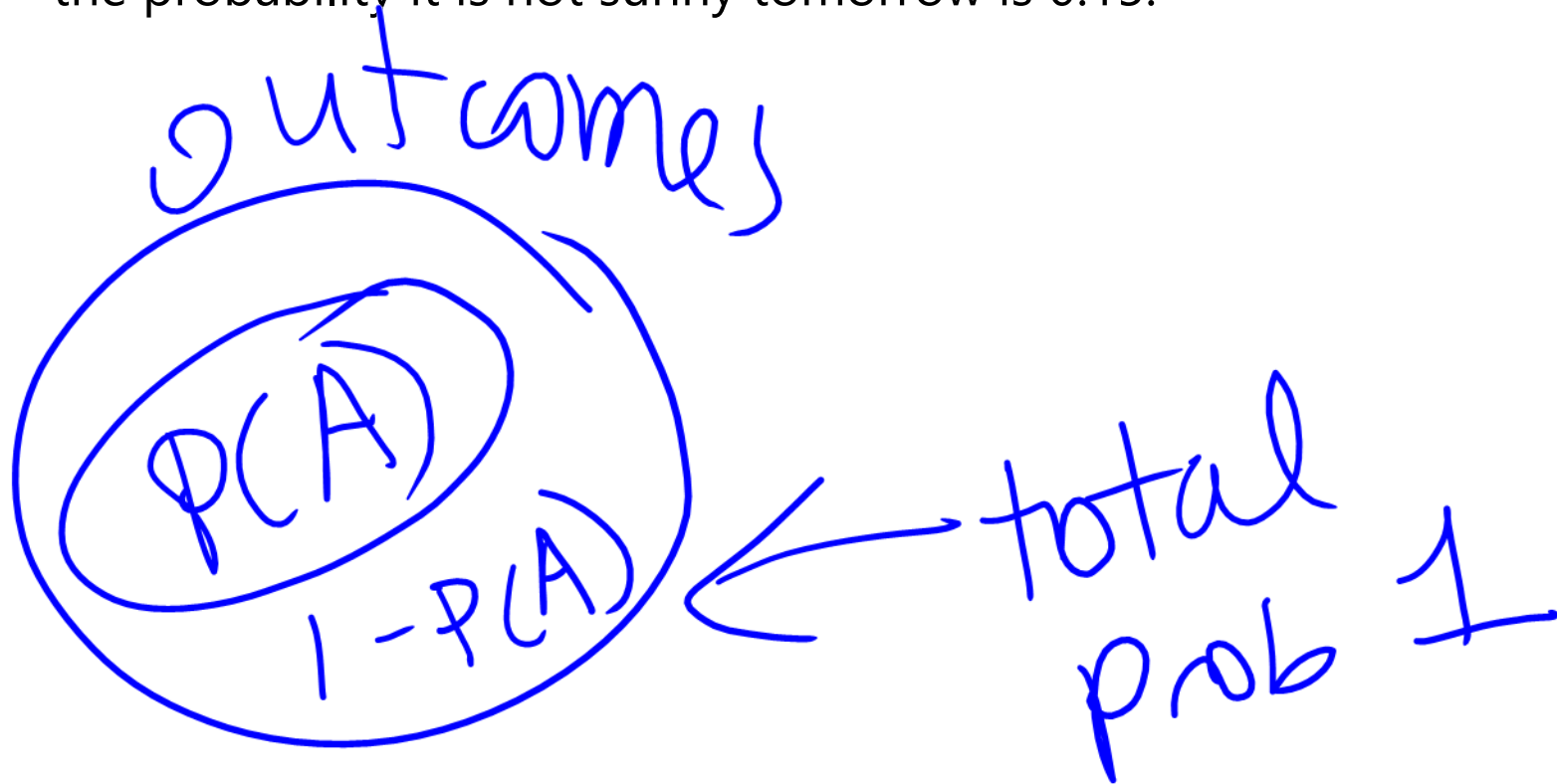
$$P(A \text{ and } B) = P(A) \cdot P(B)$$

- **Example 3:** Suppose we have a coin that is **biased**, and flips heads with probability 0.7. Each flip is independent of all other flips. We flip it 5 times. What's the probability we see 5 heads in a row?


$$\begin{aligned} & P(\text{1<sup>st</sup> flip H AND 2<sup>nd</sup> H AND ... AND 5<sup>th</sup> H}) \\ &= P(\text{1<sup>st</sup> flip H}) * P(\text{2<sup>nd</sup> H}) * \dots * P(\text{5<sup>th</sup> H}) \\ &= (0.7)^5 \end{aligned}$$

Probability that an event *doesn't* happen

- The probability that A **doesn't** happen is  $1 - P(A)$ .
- For example, if the probability it is sunny tomorrow is 0.85, then the probability it is not sunny tomorrow is 0.15.



Concept Check  – Answer at [cc.dsc10.com](https://www.cc.dsc10.com)

Every time I call my grandma , the probability that she answers her phone is  $\frac{1}{3}$ , independently for each call. If I call my grandma three times today, what is the chance that I will talk to her at least once?

- A)  $\frac{1}{3}$
- B)  $\frac{2}{3}$
- C)  $\frac{1}{2}$
- ~~D) 1~~
- E) None of the above.

$\frac{19}{27}$

8 outcomes



Y = answers (yes)  
N = doesn't (no)

$$P(\text{talk}) = 1 - P(\text{no talk}) = 1 - \left(\frac{2}{3}\right)^3$$

$$\frac{1}{3} * \frac{2}{3} * \frac{2}{3}$$

$$\left(\frac{2}{3}\right)^3$$

$$\frac{2}{3} * \frac{1}{3} * \frac{1}{3}$$

Probability of either of two events happening

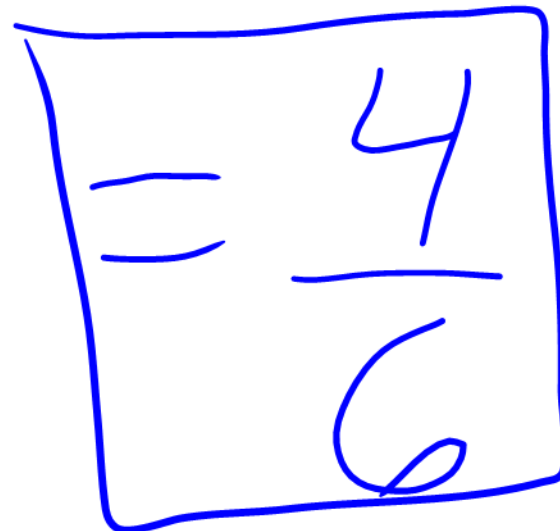


- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that either A or B occur is

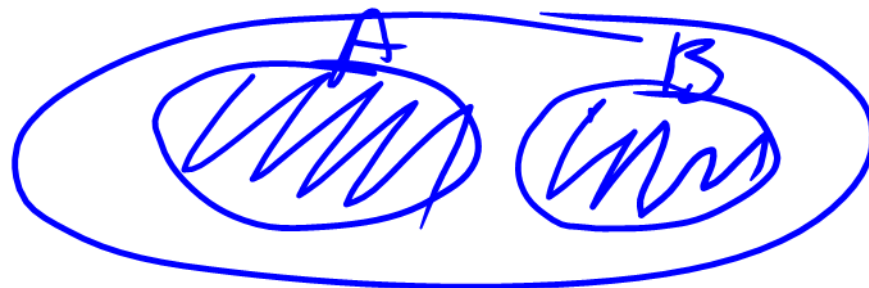
$$P(A \text{ or } B) = \frac{\text{\# of outcomes satisfying either A or B}}{\text{total \# of outcomes}}$$

- **Example 4:** I roll a fair six-sided die. What is the probability that the roll is even or at least 5?

$$\frac{3}{6} + \frac{2}{6} = \frac{5}{6} \leftarrow \text{incorrect because '6' is being counted twice}$$



← incorrect because '6' is being counted twice



The addition rule

- Suppose that if A happens, then B doesn't, and if B happens, then A doesn't.
  - Such events are called **mutually exclusive** – they have **no overlap**.
- If A and B are any two mutually exclusive events, then

$$P(A \text{ or } B) = P(A) + P(B)$$

- **Example 5:** Suppose I have two biased coins, coin A and coin B. Coin A flips heads with probability 0.6, and coin B flips heads with probability 0.3. I flip both coins once. What's the probability I see two different faces?

prob  
 $0.6 \times 0.7$   
 $\uparrow$   
 $= 0.42$

H on A AND T on B

T on A AND H on B


(mutually exclusive)

$0.42 + 0.12 = 0.54$  prob  $0.4 \times 0.3 = 0.12$

Aside: proof of the addition rule for equally-likely events

You are not required to know how to "prove" anything in this course; you may just find this interesting.

If A and B are events consisting of equally likely outcomes, and furthermore A and B are mutually exclusive (meaning they have no overlap), then

$$\begin{aligned} P(A \text{ or } B) &= \frac{\text{\# of outcomes satisfying either A or B}}{\text{total \# of outcomes}} \\ &= \frac{(\text{\# of outcomes satisfying A}) + (\text{\# of outcomes satisfying B})}{\text{total \# of outcomes}} \\ &= \frac{(\text{\# of outcomes satisfying A})}{\text{total \# of outcomes}} + \frac{(\text{\# of outcomes satisfying B})}{\text{total \# of outcomes}} \\ &= P(A) + P(B) \end{aligned}$$


## Summary, next time

- Probability describes the likelihood of an event occurring.
- There are several rules for computing probabilities. We looked at many special cases that involved equally-likely events.
- There are two general rules to be aware of:

AND

- The multiplication rule, which states that for any two events,  $P(A \text{ and } B) = P(B \text{ given } A) \cdot P(A)$ .

OR

- The addition rule, which states that for any two mutually exclusive events,  $P(A \text{ or } B) = P(A) + P(B)$ .

- **Next time:** simulations.