# Lecture 11 – Probability

DSC 10, Fall 2024

## Announcements

- Discussion is **today**. New policies starting in two weeks:
    - You must take and upload a photo with your ID for credit. No sign-in sheet option.
    - Submissions will close at **11:59PM** the day of discussion and we won't accept your photo outside of this.
- Quiz 2 is on **Wednesday** in your assigned quiz session.
    - Same time as Quiz 1, but new seats. You should get an email tomorrow with your seating assignment.
    - The quiz covers Lecture 5 through 10 and related labs and homeworks.
- Lab 3 is due **Thursday**. Homework 3 is due on **Sunday**.
    - Do as much of these assignments as possible before the quiz.

## Agenda

We'll cover the basics of probability theory. This is a math lesson; take written notes ✍🏽.

## Probability resources

Probability is a tricky subject. If it doesn't click during lecture or on the assignments, take a look at the following resources:

- **Computational and Inferential Thinking, Chapter 9.5**.
- **Theory Meets Data, Chapters 1 and 2**.
- **Khan Academy's unit on Probability**.

## Probability theory

- Some things in life *seem* random.
  - e.g., flipping a coin or rolling a die 🎲.
- The **probability** of seeing "heads" when flipping a fair coin is $\frac{1}{2}$.
- One interpretation of probability says that if we flipped a coin infinitely many times, then $\frac{1}{2}$ of the outcomes would be heads.

Terminology

- **Experiment**: A process or action whose result is random.
  - e.g., rolling a die.
  - e.g., flipping a coin twice.
- **Outcome**: The result of an experiment.
  - e.g., the possible outcomes of rolling a six-sided die are 1, 2, 3, 4, 5, and 6.
  - e.g., the possible outcomes of flipping a coin twice are HH, HT, TH, and TT.

4

- **Event**: A set of outcomes.
  - e.g., the event that the die lands on a even number is the set of outcomes {2, 4, 6}.
  - e.g., the event that the die lands on a 5 is the set of outcomes {5}.
  - e.g., the event that there is at least 1 head in 2 flips is the set of outcomes {HH, HT, TH}.

$$\frac{3}{4} = \frac{\#\text{ good}}{\#\text{ total}}$$

## Terminology

- **Probability**: A number between 0 and 1 (equivalently, between 0% and 100%) that describes the likelihood of an event.
    - 0: The event never happens.
    - 1: The event always happens.
- Notation: If $A$ is an event, $P(A)$ is the probability of that event.

## Equally-likely outcomes

- If all of the possible outcomes are equally likely, then the probability of $A$ is

$$P(A) = \frac{\#\text{ of outcomes satisfying } A}{\text{total } \#\text{ of outcomes}}$$

- **Example 1**: Suppose we flip a fair coin 3 times. What is the probability we see exactly 2 heads?

HHH TTT
HHT TTH
HTH THT
HTT THH

$\frac{3}{8}$

wrong!

0 H
1 H
2 H
3 H

$\frac{1}{4}$

Concept Check ✅ – Answer at **cc.dsc10.com**

I have three cards: red, blue, and green. What is the chance that I choose a card at random and it is green, then – **without putting it back** – I choose another card at random and it is red?

- A) $\frac{1}{9}$
- B) $\frac{1}{6}$
- C) $\frac{1}{3}$
- D) $\frac{2}{3}$
- E) None of the above.

P(first G) * P(second R given first G)

R B G

$\frac{1}{3}$ * $\frac{1}{2}$

R B

$= \frac{1}{6}$

elementary solution

RB GB BR

RG GR BG

$\boxed{\frac{1}{6}}$

like prob. of $B$, except out of just $A$

## Conditional probabilities

- Two events $A$ and $B$ can both happen. Suppose that we know $A$ has happened, but we don't know if $B$ has.
- If all outcomes are equally likely, then the conditional probability of $B$ given $A$ is:

$B|A$

$$P(B \text{ given } A) = \frac{\# \text{ of outcomes satisfying both } A \text{ and } B}{\# \text{ of outcomes satisfying } A}$$

- Intuitively, this is similar to the definition of the regular probability of $B$, $P(B) = \frac{\# \text{ of outcomes satisfying } B}{\text{total } \# \text{ of outcomes}}$, if you restrict the set of possible outcomes to be just those in event $A$.


Problem reduces to: how much of A is taken up by B

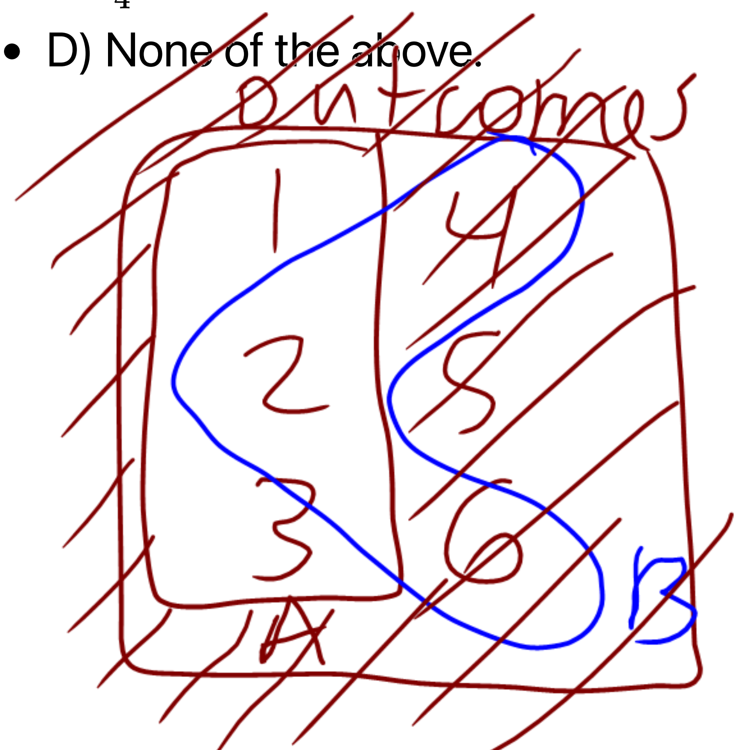
## Concept Check ✅ – Answer at **cc.dsc10.com**

$$P(B \text{ given } A) = \frac{\# \text{ of outcomes satisfying both } A \text{ and } B}{\# \text{ of outcomes satisfying } A}$$

I roll a six-sided die and don't tell you what the result is, but I tell you that it is 3 or less. What is the probability that the result is even?

A  B

- A) $\frac{1}{2}$
- B) $\frac{1}{3}$
- C) $\frac{1}{4}$
- D) None of the above.




$P(\underbrace{\text{even}}_{B} \text{ given } \underbrace{\le 3}_{A})$

$\frac{1}{3}$

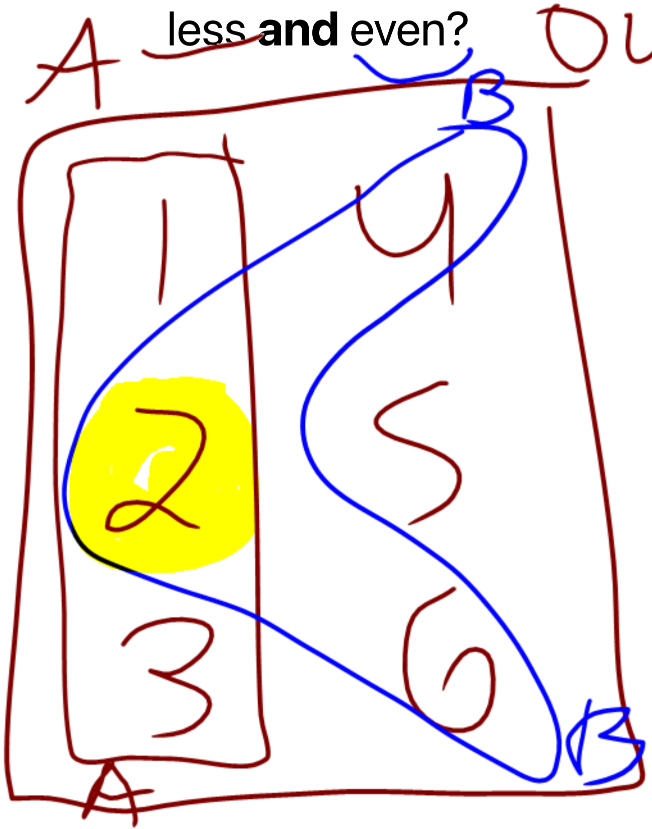
Probability that two events both happen

- Suppose again that $A$ and $B$ are two events, and that all outcomes are equally likely. Then, the probability that both $A$ and $B$ occur is

$$P(A \text{ and } B) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\text{total } \#\text{ of outcomes}}$$

- **Example 2**: I roll a fair six-sided die. What is the probability that the roll is 3 or less **and** even?

outcomes




$$\frac{1}{6}$$

both

make A happen

make B happen given that A happened

The multiplication rule

- The multiplication rule specifies how to compute the probability of both $A$ and $B$ happening, even if all outcomes are not equally likely.

$$\underbrace{P(A \text{ and } B)}_{\text{both}} = \underbrace{P(A)}_{\text{make A happen}} \cdot \underbrace{P(B \text{ given } A)}_{\text{make B happen given that A happened}}$$

- **Example 2, again**: I roll a fair six-sided die. What is the probability that the roll is $\underbrace{\text{3 or less}}_{A}$ and $\underbrace{\text{even}}_{B}$?

$$P(\leq 3 \text{ and even}) = P(\leq 3) \times P(\text{even given} \leq 3)$$

$$= \frac{\cancel{3}}{6} \times \frac{1}{\cancel{3}}$$

$$= \frac{1}{6}$$

| | |
|---|---|
| 1 | 4 |
| 2 | 5 |
| 3 | 6 |

What if $A$ isn't affected by $B$? 🤔

- The multiplication rule states that, for any two events $A$ and $B$,

$$P(A \text{ and } B) = P(A) \cdot \boxed{P(B \text{ given } A)}$$

- What if knowing that $A$ happens doesn't tell you anything about the likelihood of $B$ happening?
  - Suppose we flip a fair coin three times.
  - The probability that the second flip is heads doesn't depend on the result of the first flip.
- Then, what is $P(A \text{ and } B)$?

$$\underbrace{P(A \text{ and } B) = P(A) \cdot P(B)}_{\text{when } A \text{ and } B \text{ are independent, and } = *}$$

when $A$ is irrelevant to $B$

Independent events

and = $\times$

- Two events $A$ and $B$ are independent if $P(B \text{ given } A) = P(B)$, or equivalently if

special case

$$\underline{P(A \text{ and } B)} = P(A) \cdot P(B)$$

not

$\frac{1}{32}$

because some outcomes more likely than others

- **Example 3**: Suppose we have a coin that is **biased**, and flips heads with probability 0.7. Each flip is independent of all other flips. We flip it 5 times. What's the probability we see 5 heads in a row?

$P\left(\begin{matrix}\text{first}\\ H\end{matrix} \text{ and } \begin{matrix}\text{second}\\ H\end{matrix} \text{ and } \ldots \begin{matrix}\text{fifth}\\ H\end{matrix}\right)$

$P\left(\begin{matrix}\text{first}\\ H\end{matrix}\right) \times P\left(\begin{matrix}\text{second}\\ H\end{matrix}\right) \times \ldots \times P\left(\begin{matrix}\text{fifth}\\ H\end{matrix}\right)$

$0.7 \times 0.7 \times \ldots \times 0.7$

$= \boxed{0.7^5}$

## Probability that an event *doesn't* happen

- The probability that $A$ **doesn't** happen is $1 - P(A)$ .
- For example, if the probability it is sunny tomorrow is 0.85, then the probability it is not sunny tomorrow is 0.15.

$$P(NNN) = P(N)^3 = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

Concept Check ✅ – Answer at **cc.dsc10.com**

Every time I call my grandma 👵, the probability that she answers her phone is $\frac{1}{3}$, independently for each call. If I call my grandma three times today, what is the chance that I will talk to her at least once?

- ~~A) $\frac{1}{3}$~~
- B) $\frac{2}{3}$
- C) $\frac{1}{2}$
- ~~D) 1~~
- E) None of the above.

$$1 - \frac{8}{27} = \boxed{\frac{19}{27}}$$

YYY NYY
YNN NYN
YYN NNY
YNY NNN

if you were to list out all outcomes:

YNN $\rightarrow P(YNN) = P(Y) \times P(N) \times P(N)$

independent

$= \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3}$

how many outcomes? 8

all equally likely? NO

not i $\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$

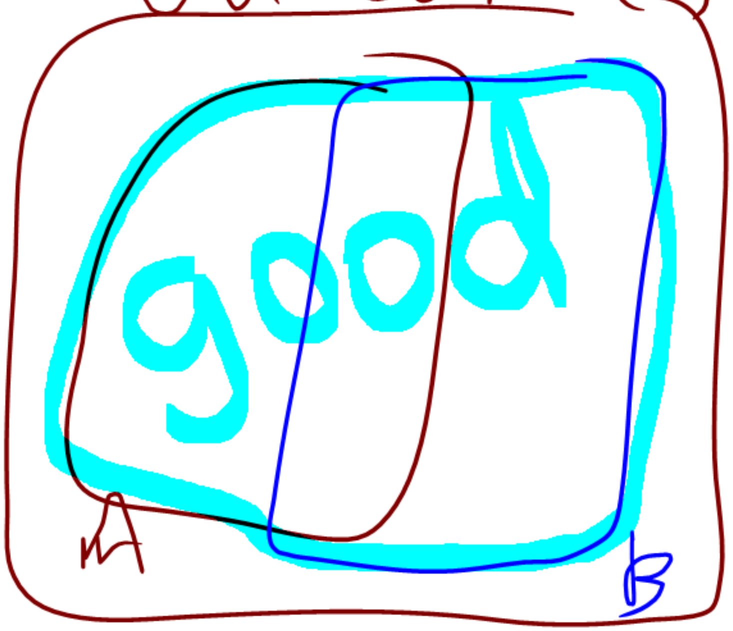
Probability of either of two events happening

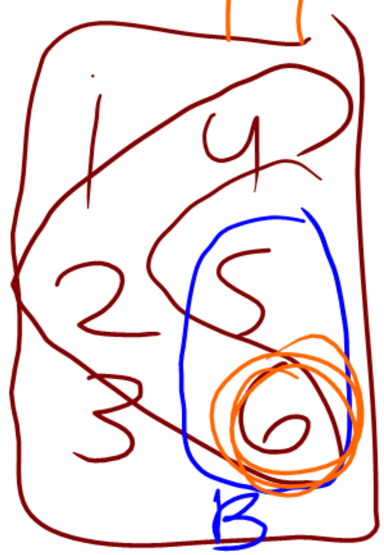
- Suppose again that $A$ and $B$ are two events, and that all outcomes are equally likely. Then, the probability that either $A$ or $B$ occur is

$$P(A \text{ or } B) = \frac{\#\text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \#\text{ of outcomes}}$$

- **Example 4**: I roll a fair six-sided die. What is the probability that the roll is even (A) or at least 5? (B)

outcomes




$= \frac{4}{6} = \frac{2}{3}$


## The addition rule

special case

- Suppose that if $A$ happens, then $B$ doesn't, and if $B$ happens, then $A$ doesn't.
  - Such events are called **mutually exclusive** – they have **no overlap**.
- If $A$ and $B$ are any two mutually exclusive events, then

$$P(A \text{ or } B) = P(A) + P(B)$$

or = + when mutually exclusive

- **Example 5**: Suppose I have two biased coins, coin $A$ and coin $B$. Coin $A$ flips heads with probability 0.6, and coin $B$ flips heads with probability 0.3. I flip both coins once. What's the probability I see two different faces?

application: cases that don't overlap

case 1: $A \Rightarrow H$, $B \Rightarrow T$

case 2: $A \Rightarrow T$, $B \Rightarrow H$

$P(\text{case 1 or case 2}) = P(\text{case 1}) + P(\text{case 2})$

$P(\text{case 1}) = P(A \text{ is } H \text{ and } B \text{ is } T) = P(A \text{ is } H) * P(B \text{ is } T) = 0.6 * 0.7$

$P(\text{case 2}) = 0.4 * 0.3$

$0.42 + 0.12 = 0.54$

## Aside: Proof of the addition rule for equally-likely events

You are not required to know how to "prove" anything in this course; you may just find this interesting.

If $A$ and $B$ are events consisting of equally likely outcomes, and furthermore $A$ and $B$ are mutually exclusive (meaning they have no overlap), then

$$
\begin{aligned}
P(A \text{ or } B) &= \frac{\# \text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \# \text{ of outcomes}} \\
&= \frac{(\# \text{ of outcomes satisfying } A) + (\# \text{ of outcomes satisfying } B)}{\text{total } \# \text{ of outcomes}} \\
&= \frac{(\# \text{ of outcomes satisfying } A)}{\text{total } \# \text{ of outcomes}} + \frac{(\# \text{ of outcomes satisfying } B)}{\text{total } \# \text{ of outcomes}} \\
&= P(A) + P(B)
\end{aligned}
$$

## Summary, next time

- Probability describes the likelihood of an event occurring.
- There are several rules for computing probabilities. We looked at many special cases that involved equally-likely events.
- There are two general rules to be aware of:
    - The **multiplication rule**, which states that for any two events, $P(A \text{ and } B) = P(B \text{ given } A) \cdot P(A)$ .
    - The **addition rule**, which states that for any two **mutually exclusive** events, $P(A \text{ or } B) = P(A) + P(B)$.
- **Next time:** Simulations.