Lecture 11 – Probability

DSC 10, Spring 2024

Announcements

- Discussion is this afternoon. Problems are **here**.
- Lab 3 is due **tomorrow at 11:59PM**.
- Quiz 2 is on **Friday** in your assigned quiz session.
    - You should get an email tomorrow with your seating assignment.
    - Bring your ID and a pencil.
    - This is a 20 minute paper-based quiz with no aids allowed.
    - The quiz covers Lecture 5 through 9 and related labs and homeworks.
    - Quiz 2 is **more challenging** than Quiz 1, and next week's Midterm Exam will be more challenging than Quiz 2. 📈
- Homework 3 is due on **Tuesday**.
    - Problems 1 and 2 only are relevant to Quiz 2.

Agenda

We'll cover the basics of probability theory. This is a math lesson; take written notes ✍🏽.

Probability resources

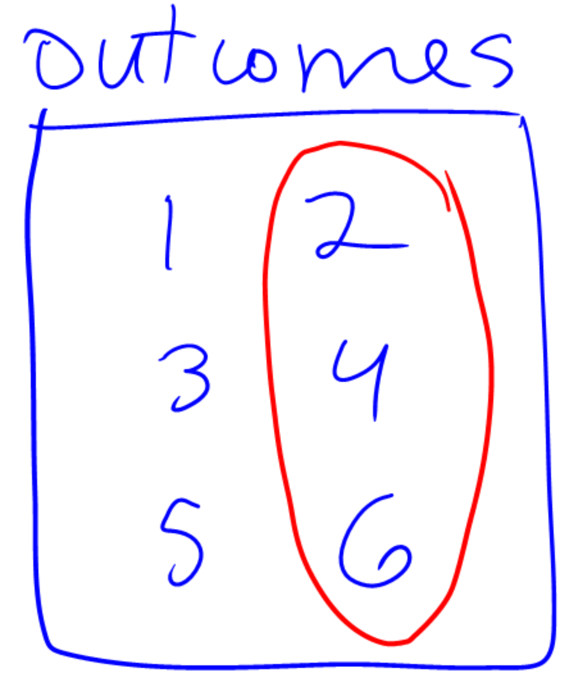
Probability is a tricky subject. If it doesn't click during lecture or on the assignments, take a look at the following resources:

- **Computational and Inferential Thinking, Chapter 9.5**.
- **Theory Meets Data, Chapters 1 and 2**.
- **Khan Academy's unit on Probability**.

Probability theory

- Some things in life *seem* random.
    - e.g., flipping a coin or rolling a die 🎲.
- The **probability** of seeing "heads" when flipping a fair coin is $\frac{1}{2}$.
- One interpretation of probability says that if we flipped a coin infinitely many times, then $\frac{1}{2}$ of the outcomes would be heads.

Terminology



- **Experiment**: A process or action whose result is random.
    - e.g., rolling a die.
    - e.g., flipping a coin twice.
- **Outcome**: The result of an experiment.
    - e.g., the possible outcomes of rolling a six-sided die are 1, 2, 3, 4, 5, and 6.
    - e.g., the possible outcomes of flipping a coin twice are HH, HT, TH, and TT.
- **Event**: A set of outcomes.
    - e.g., the event that the die lands on a even number is the set of outcomes {2, 4, 6}.
    - e.g., the event that the die lands on a 5 is the set of outcomes {5}.
    - e.g., the event that there is at least 1 head in 2 flips is the set of outcomes {HH, HT, TH}.

Terminology

- **Probability**: A number between 0 and 1 (equivalently, between 0% and 100%) that describes the likelihood of an event.
    - 0: The event never happens.
    - 1: The event always happens.
- Notation: If A is an event, $P(A)$ is the probability of that event.

$P(\text{rolling even}) = \frac{3}{6}$

Equally-likely outcomes

- If all of the possible outcomes are equally likely, then the probability of A is

$$P(A) = \frac{\#\text{ of outcomes satisfying } A}{\text{total } \#\text{ of outcomes}} \rightarrow \frac{\#\text{ good}}{\#\text{ outcomes}}$$

1	2
3	4
5	6

- **Example 1**: Suppose we flip a fair coin 3 times. What is the probability we see exactly 2 heads?

outcomes

HHH	THH
HHT	THT
HTH	TTH
HTT	TTT

$\frac{3}{8}$

another way

outcomes

0H, 3T
1H, 2T
2H, 1T
3H, 0T

~~$\frac{1}{4}$~~

different

not all outcomes are equally likely

R B ~~G~~

Concept Check ✅ – Answer at **cc.dsc10.com**

I have three cards: red, blue, and green. What is the chance that I choose a card at random and it is green, then **– without putting it back –** I choose another card at random and it is red?

without replacement

- A) $\frac{1}{9}$
- B) $\frac{1}{6}$
- C) $\frac{1}{3}$
- D) $\frac{2}{3}$
- E) None of the above.

outcomes

RB RG
GB GR
BG BR

$\frac{1}{6}$

another way: (multiplication rule)

P(green on 1st card) $= \frac{1}{3}$

P(red on 2nd card if got green on 1st card) $= \frac{1}{2}$

$\Rightarrow$ multiply $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

$P(B) = \frac{10}{21}$ $\qquad$ $P(B \text{ given } A) = \frac{4}{12} = \frac{1}{3}$

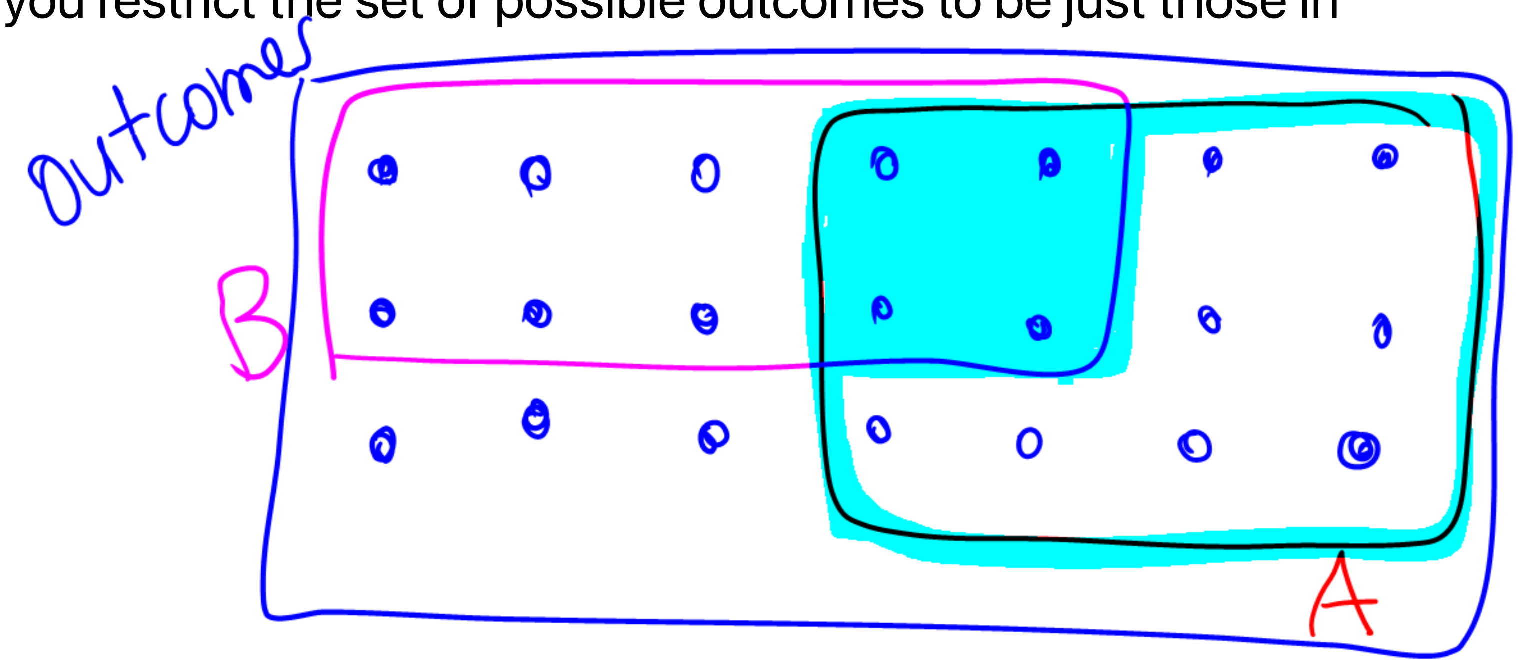
Conditional probabilities

- Two events A and B can both happen. Suppose that we know A has happened, but we don't know if B has.
- If all outcomes are equally likely, then the conditional probability of B given A is:

$$\boxed{P(B \text{ given } A) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\#\text{ of outcomes satisfying } A}}$$

one third of A is taken up by B

- Intuitively, this is similar to the definition of the regular probability of B, $P(B) = \frac{\#\text{ of outcomes satisfying } B}{\text{total }\#\text{ of outcomes}}$, if you restrict the set of possible outcomes to be just those in event A.



Knowledge that A happened means set of possible outcomes is just A

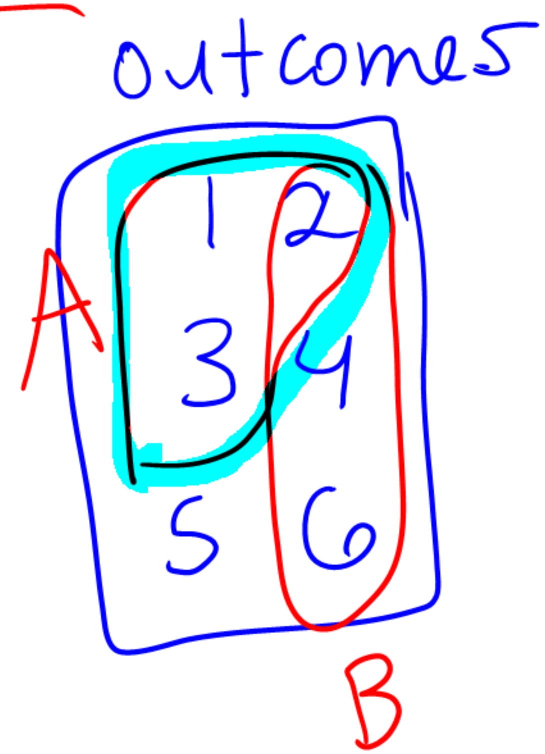
Concept Check ✅ – Answer at **cc.dsc10.com**

$$P(B \text{ given } A) = \frac{\# \text{ of outcomes satisfying both } A \text{ and } B}{\# \text{ of outcomes satisfying } A}$$

I roll a six-sided die and don't tell you what the result is, but I tell you that it is 3 or less. What is the probability that the result is even?

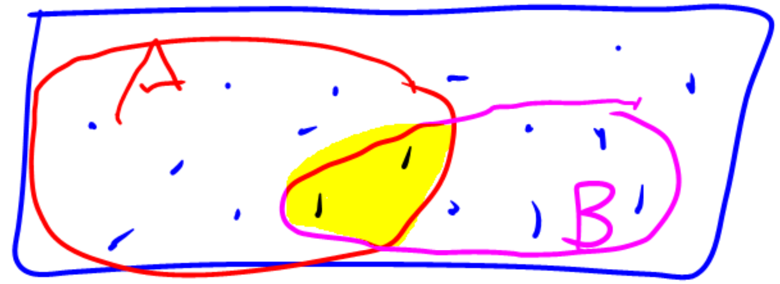
A

- A) $\frac{1}{2}$
- B) $\frac{1}{3}$
- C) $\frac{1}{4}$
- D) None of the above.



$\frac{1}{3}$

Probability that two events both happen



- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that both A and B occur is

$$P(A \text{ and } B) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\text{total } \#\text{ of outcomes}}$$

- **Example 2**: I roll a fair six-sided die. What is the probability that the roll is 3 or less **and** even?

outcomes

1	2
3	4
5	6

$\frac{1}{6}$

The multiplication rule

- The multiplication rule specifies how to compute the probability of both A and B happening, even if all outcomes are not equally likely.

for A and B to both happen, first make A happen

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

then make B happen in the case A happened

- **Example 2, again**: I roll a fair six-sided die. What is the probability that the roll is 3 or less and even?

$P(\leq 3 \text{ and even}) =$

$P(\leq 3) * P(\text{even given} \leq 3)$

$\frac{3}{6} * \frac{1}{3}$

$= \frac{1}{6}$

cards

$P(G\ 1^{st} \text{ and } R\ 2^{nd})$

$= P(G\ 1^{st}) * P(R\ 2^{nd} \text{ given } G\ 1^{st})$

$= \frac{1}{3} * \frac{1}{2}$

What if A isn't affected by B? 🤔

- The multiplication rule states that, for any two events A and B,

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

for independent A and B, this part simplifies to just P(B)

- What if knowing that A happens doesn't tell you anything about the likelihood of B happening?
  - Suppose we flip a fair coin three times.
  - The probability that the second flip is heads doesn't depend on the result of the first flip.
- Then, what is $P(A \text{ and } B)$?

Independent events

ind: knowledge of A doesn't provide any knowledge of B

- Two events A and B are independent if $P(B \text{ given } A) = P(B)$, or equivalently if

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

- **Example 3**: Suppose we have a coin that is **biased**, and flips heads with probability 0.7. Each flip is independent of all other flips. We flip it 5 times. What's the probability we see 5 heads in a row?

$P(\text{H on 1st flip AND H on 2nd flip AND} \ldots)$

$= P(\text{H on 1st flip}) * P(\text{H on 2nd flip}) * \ldots$

$= 0.7 * 0.7 * \ldots = (0.7)^5$

(not 0.7 times 5)

ex. roll die, events ≤ 2, even are independent

1	2
3	4
5	6

$P(\text{even given } \leq 2) = \frac{1}{2}$

$P(\text{even}) = \frac{1}{2}$

ex. ≤ 3, even are dependent

1	2
3	4
5	6

$P(\text{even given } \leq 3) = \frac{1}{3}$

$P(\text{even}) = \frac{1}{2}$

Probability that an event *doesn't* happen Complement rule

- The probability that A **doesn't** happen is $1 - P(A)$.
- For example, if the probability it is sunny tomorrow is 0.85, then the probability it is not sunny tomorrow is 0.15.

N = no
Y = yes

outcomes

NNN
YYN
NYN ...

8 outcomes but NOT all equally likely

Concept Check ✅ – Answer at **cc.dsc10.com**

Every time I call my grandma 👵, the probability that she answers her phone is $\frac{1}{3}$, independently for each call. If I call my grandma three times today, what is the chance that I will talk to her at least once?

- ~~A) $\frac{1}{3}$~~ must be $> 1/3$ with multiple attempts
- B) $\frac{2}{3}$
- C) $\frac{1}{2}$
- ~~D) 1~~ must be < 1 because not guaranteed
- E) None of the above.

$\neq \frac{7}{8}$

$$P(\text{no answer}) = \frac{2}{3}$$

$$P(\text{no answer on 1st AND no answer on 2nd AND no answer on 3rd}) = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

$$P(\text{answer at least once}) = 1 - P(\text{never answers}) = 1 - \frac{8}{27} = \boxed{\frac{19}{27}}$$

Probability of either of two events happening

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that either A or B occur is

$$P(A \text{ or } B) = \frac{\#\text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \#\text{ of outcomes}}$$

- **Example 4**: I roll a fair six-sided die. What is the probability that the roll is even or at least 5?

outcomes

$\frac{4}{6} = \frac{2}{3}$

wrong: $\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$

double counting

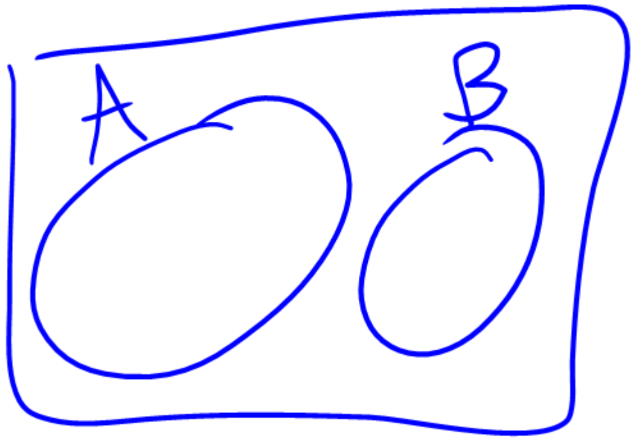
the overlap when adding

The addition rule

- Suppose that if A happens, then B doesn't, and if B happens, then A doesn't.
    - Such events are called **mutually exclusive** – they have **no overlap**.
- If A and B are any two mutually exclusive events, then

$$P(A \text{ or } B) = P(A) + P(B)$$

- **Example 5**: Suppose I have two biased coins, coin A and coin B. Coin A flips heads with probability 0.6, and coin B flips heads with probability 0.3. I flip both coins once. What's the probability I see two different faces?



$$P\left(\underbrace{(H \text{ on } A \text{ and } T \text{ on } B)}\ \text{OR}\ (T \text{ on } A \text{ and } H \text{ on } B)\right)$$

$$0.6 * 0.7 + 0.4 * 0.3$$

Aside: Proof of the addition rule for equally-likely events

You are not required to know how to "prove" anything in this course; you may just find this interesting.

If A and B are events consisting of equally likely outcomes, and furthermore A and B are mutually exclusive (meaning they have no overlap), then

$$\begin{aligned}
P(A \text{ or } B) &= \frac{\#\text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \# \text{ of outcomes}} \\
&= \frac{(\#\text{ of outcomes satisfying } A) + (\#\text{ of outcomes satisfying } B)}{\text{total } \# \text{ of outcomes}} \\
&= \frac{(\#\text{ of outcomes satisfying } A)}{\text{total } \# \text{ of outcomes}} + \frac{(\#\text{ of outcomes satisfying } B)}{\text{total } \# \text{ of outcomes}} \\
&= P(A) + P(B)
\end{aligned}$$

Summary, next time

- Probability describes the likelihood of an event occurring.
- There are several rules for computing probabilities. We looked at many special cases that involved equally-likely events.
- There are two general rules to be aware of:
    - The **multiplication rule**, which states that for any two events,

      $P(A \text{ and } B) = P(B \text{ given } A) \cdot P(A)$.
    - The **addition rule**, which states that for any two **mutually exclusive** events,

      $P(A \text{ or } B) = P(A) + P(B)$.
- **Next time:** Simulations.