Lecture 11 – Probability

DSC 10, Summer 2024

Announcements

- We have a ***very*** busy week! Lots of resources: Ed, office hours, discussion, midterm review.
- Lab 3 is due tonight.
- Midterm Exam is Thursday 11AM in this room. Review in discussion today and tomorrow in lab session.
- Homework 3 due Friday.
- Midterm project due Monday.

Agenda

We'll cover the basics of probability theory. This is a math lesson; take written notes ✍🏽.

Probability resources

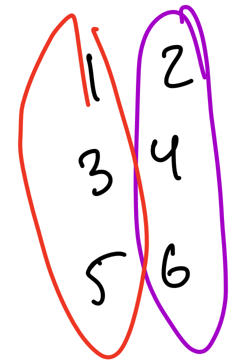
Probability is a tricky subject. If it doesn't click during lecture or on the assignments, take a look at the following resources:

- **Computational and Inferential Thinking, Chapter 9.5**.
- **Theory Meets Data, Chapters 1 and 2**.
- **Khan Academy's unit on Probability**.

Probability theory

- Some things in life *seem* random.
    - e.g., flipping a coin or rolling a die 🎲.
- The **probability** of seeing "heads" when flipping a fair coin is $\frac{1}{2}$.
- One interpretation of probability says that if we flipped a coin infinitely many times, then $\frac{1}{2}$ of the outcomes would be heads.

Terminology



- **Experiment**: A process or action whose result is random.
  - e.g., rolling a die.
  - e.g., flipping a coin twice.
- **Outcome**: The result of an experiment.
  - e.g., the possible outcomes of rolling a six-sided die are 1, 2, 3, 4, 5, and 6.
  - e.g., the possible outcomes of flipping a coin twice are HH, HT, TH, and TT.
- **Event**: A set of outcomes.
  - e.g., the event that the die lands on a even number is the set of outcomes {2, 4, 6}.
  - e.g., the event that the die lands on a 5 is the set of outcomes {5}.
  - e.g., the event that there is at least 1 head in 2 flips is the set of outcomes {HH, HT, TH}.

Terminology

- **Probability**: A number between 0 and 1 (equivalently, between 0% and 100%) that describes the likelihood of an event.
    - 0: The event never happens.
    - 1: The event always happens.
- Notation: If A is an event, $P(A)$ is the probability of that event.

fair coin : $P(\text{Heads}) = 1/2$

Equally-likely outcomes

- If all of the possible outcomes are equally likely, then the probability of A is

$$P(A) = \boxed{\frac{\#\text{ of outcomes satisfying } A}{\text{total } \#\text{ of outcomes}}}$$

- **Example 1**: Suppose we flip a fair coin 3 times. What is the probability we see exactly 2 heads?

$P(\text{rolling even number}) = \frac{3}{6} = \frac{1}{2}$

die

1 2
3 4
5 6

outcomes of 3 flips

HHH THH
HHT THT
HTH TTH
HTT TTT

$P(\text{exactly 2 heads}) = \frac{3}{8}$

Wrong!

outcomes

0 H 1 H
2 H 3 H

$P(2H) = \frac{1}{4}$

Wrong! outcomes are not equally likely

Concept Check ✅ – Answer at **cc.dsc10.com**

I have three cards: red, blue, and green. What is the chance that I choose a card at random and it is green, then – **without putting it back** – I choose another card at random and it is red?

- A) $\frac{1}{9}$
- B) $\frac{1}{6}$? majority
- C) $\frac{1}{3}$
- D) $\frac{2}{3}$
- E) None of the above.

3 cards, 1/3 of green

$P(GR) = \frac{1}{3} \times \frac{1}{2}$

then 2 cards, 1/2 red

Outcomes

RB BG GB

RG BR $\boxed{GR}$

$\frac{1}{6}$

Conditional probabilities

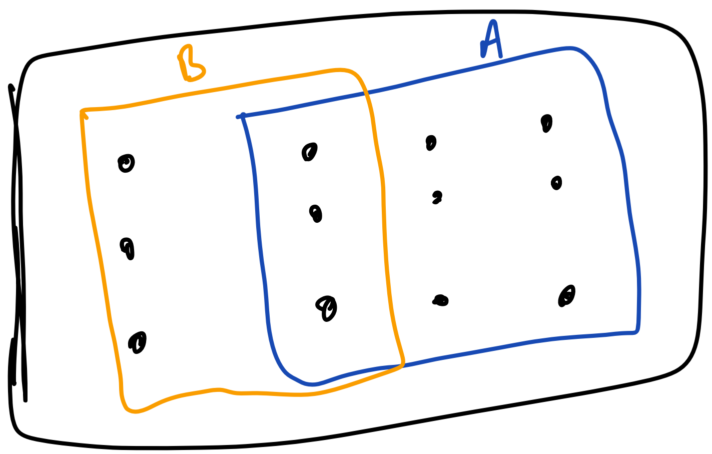
- Two events A and B can both happen. Suppose that we know A has happened, but we don't know if B has.
- If all outcomes are equally likely, then the conditional probability of B given A is:

$P(B|A)$

$$P(B \text{ given } A) = \frac{\# \text{ of outcomes satisfying both } A \text{ and } B}{\# \text{ of outcomes satisfying } A}$$

- Intuitively, this is similar to the definition of the regular probability of B, $P(B) = \frac{\# \text{ of outcomes satisfying } B}{\text{total } \# \text{ of outcomes}}$, if you restrict the set of possible outcomes to be just those in event A.

limit options to just A



$P(B \text{ given } A) = \frac{3}{9} = \frac{1}{3}$

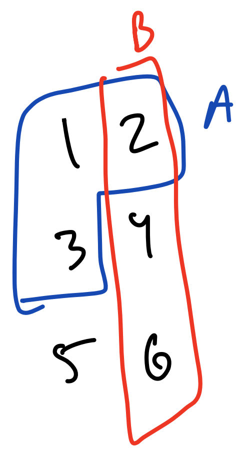
Concept Check ✅ – Answer at **cc.dsc10.com**

$$P(B \text{ given } A) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\#\text{ of outcomes satisfying } A}$$

I roll a six-sided die and don't tell you what the result is, but I tell you that it is 3 or less. What is the probability that the result is even?

- A) $\frac{1}{2}$
- B) $\frac{1}{3}$
- C) $\frac{1}{4}$
- D) None of the above.

given A → less than or equal to 3



$\boxed{\frac{1}{3}}$ → 2 (even) / 3 or less

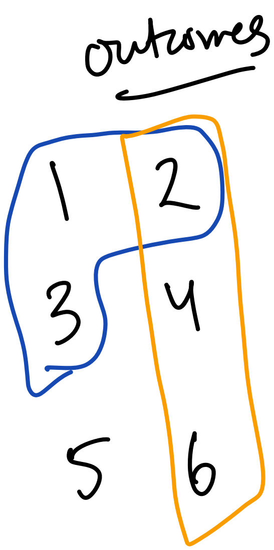
Probability that two events both happen

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that both A and B occur is

$$P(A \text{ and } B) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\text{total } \#\text{ of outcomes}}$$

no conditions or prior knowledge

- **Example 2**: I roll a fair six-sided die. What is the probability that the roll is 3 or less **and** even?



$$P(3 \text{ or less AND even}) = \frac{1}{6}$$

The multiplication rule

- The multiplication rule specifies how to compute the probability of both A and B happening, even if all outcomes are not equally likely.

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

- **Example 2, again**: I roll a fair six-sided die. What is the probability that the roll is 3 or less and even?

$$P(\leq 3 \text{ and even}) = P(\leq 3) \cdot P(\text{even given } \leq 3)$$

$$= \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

$$= P(\text{even}) \cdot P(\leq 3 \text{ given even})$$

$$= \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

1 2
3 4
5 6

What if A isn't affected by B? 🤔

- The multiplication rule states that, for any two events A and B,

$$P(A \text{ and } B) = P(A) \cdot \overbrace{P(B \text{ given } A)}^{P(B)}$$

if A & B are independent

- What if knowing that A happens doesn't tell you anything about the likelihood of B happening?
    - Suppose we flip a fair coin three times.
    - The probability that the second flip is heads doesn't depend on the result of the first flip.
- Then, what is $P(A \text{ and } B)$?

$P(A \text{ given } B) = P(A)$ if independent

Independent: roll ≤ 2

roll even

Independent events

- Two events A and B are independent if $P(B \text{ given } A) = P(B)$, or equivalently if

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

- **Example 3**: Suppose we have a coin that is **biased**, and flips heads with probability 0.7. Each flip is independent of all other flips. We flip it 5 times. What's the probability we see 5 heads in a row?

$$P(HHHHH) = P(H \text{ 1st and } H \text{ 2nd and} \ldots)$$
$$= P(H \text{ 1st}) \cdot P(H \text{ 2nd}) \cdot \ldots \cdot P(H \text{ 5th})$$
$$= (0.7)(0.7)(0.7)(0.7)(0.7)$$
$$= (0.7)^5$$

NOT $(0.7) \cdot 5 \to \cancel{3.5}$

Probability that an event *doesn’t* happen

- The probability that A **doesn't** happen is $1 - P(A)$.
- For example, if the probability it is sunny tomorrow is 0.85, then the probability it is not sunny tomorrow is 0.15.

Concept Check ✅ – Answer at **cc.dsc10.com**

Every time I call my grandma 👵, the probability that she answers her phone is $\frac{1}{3}$, independently for each call. If I call my grandma three times today, what is the chance that I will talk to her at least once?

- A) $\frac{1}{3}$
- B) $\frac{2}{3}$
- C) $\frac{1}{2}$
- D) 1
- E) None of the above.

Outcomes

NNN YNN
NNY YNY
NYN YYN
NYY YYY

$P(NNN) = \left(\frac{2}{3}\right)^3$

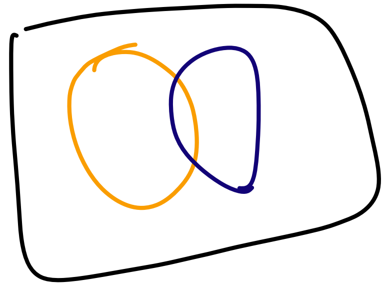
$P(YYY) = \left(\frac{1}{3}\right)^3$

$P(\text{at least } 1) = 1 - P(\text{no calls picked up})$

$1 - \left(\frac{2}{3}\right)^3$

$= 1 - \frac{8}{27} = \boxed{\frac{19}{27}}$

Probability of either of two events happening



- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that either A or B occur is

$$P(A \text{ or } B) = \frac{\# \text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \# \text{ of outcomes}}$$

- **Example 4**: I roll a fair six-sided die. What is the probability that the roll is even or at least 5?

1	2
3	4
5	6

$\{2, 4, 5, 6\}$

$$\frac{4}{6} = \frac{2}{3}$$

wrong $P(A \text{ or } B) = \underbrace{P(A) + P(B)}_{\text{NOT generally true}}$

$= P(\text{even}) + P(\geq 5)$

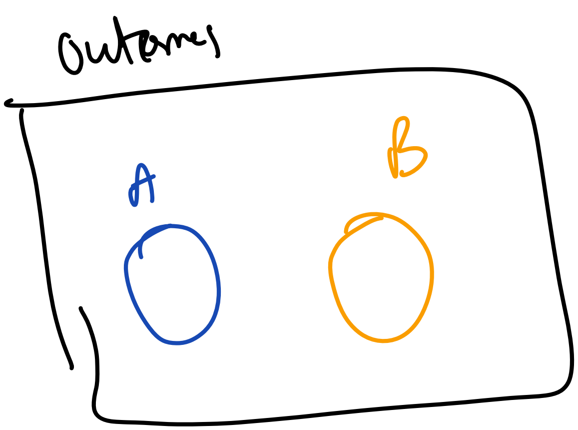
$= \frac{3}{6} + \frac{2}{6} = 5/6$ X wrong

The addition rule

- Suppose that if A happens, then B doesn't, and if B happens, then A doesn't.
    - Such events are called **mutually exclusive** – they have **no overlap**.
- If A and B are any two mutually exclusive events, then

$$P(A \text{ or } B) = P(A) + P(B)$$

- **Example 5**: Suppose I have two biased coins, coin A and coin B. Coin A flips heads with probability 0.6, and coin B flips heads with probability 0.3. I flip both coins once. What's the probability I see two different faces?



$P(\text{two diff faces}) =$

$P(A\,H \text{ and } B\,T)$ OR $P(A\,T \text{ and } B\,H)$

$= (0.6)(0.7) + (0.4)(0.3)$

Aside: Proof of the addition rule for equally-likely events

You are not required to know how to "prove" anything in this course; you may just find this interesting.

If A and B are events consisting of equally likely outcomes, and furthermore A and B are mutually exclusive (meaning they have no overlap), then

$$\begin{aligned} P(A \text{ or } B) &= \frac{\# \text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \# \text{ of outcomes}} \\ &= \frac{(\# \text{ of outcomes satisfying } A) + (\# \text{ of outcomes satisfying } B)}{\text{total } \# \text{ of outcomes}} \\ &= \frac{(\# \text{ of outcomes satisfying } A)}{\text{total } \# \text{ of outcomes}} + \frac{(\# \text{ of outcomes satisfying } B)}{\text{total } \# \text{ of outcomes}} \\ &= P(A) + P(B) \end{aligned}$$

Summary, next time

- Probability describes the likelihood of an event occurring.
- There are several rules for computing probabilities. We looked at many special cases that involved equally-likely events.
- There are two general rules to be aware of:
    - The **multiplication rule**, which states that for any two events, $P(A \text{ and } B) = P(B \text{ given } A) \cdot P(A)$.
    - The **addition rule**, which states that for any two **mutually exclusive** events, $P(A \text{ or } B) = P(A) + P(B)$.
- **Next time:** Simulations.