# Lecture 11 – Probability

DSC 10, Winter 2024

# Announcements

- Extra practice session is tonight. Problems are **here**.
  - This is the best way to prepare for the next quiz.
- Lab 3 is due **tomorrow at 11:59PM**.
- Quiz 3 is on **Monday in discussion**.
  - It covers lectures 8 through 11, which includes today.
- Homework 3 is due on **Thursday at 11:59PM**.

## Agenda

We'll cover the basics of probability theory. This is a math lesson; take written notes ✍🏽.

## Probability resources

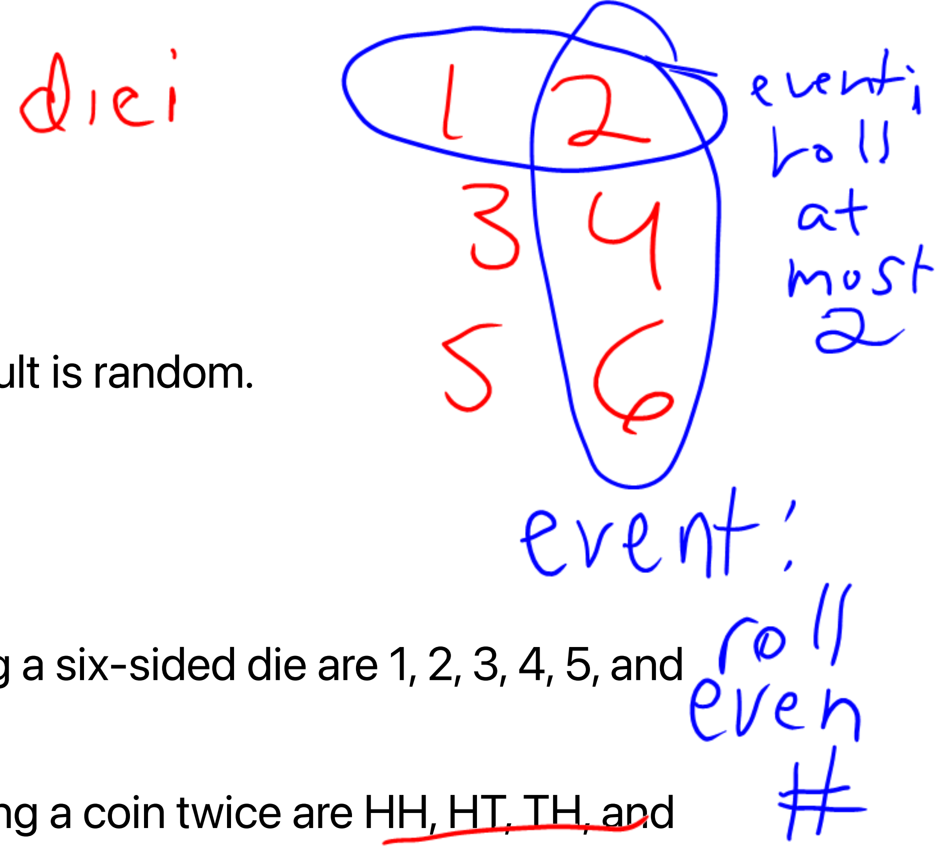
Probability is a tricky subject. If it doesn't click during lecture or on the assignments, take a look at the following resources:

- **Computational and Inferential Thinking, Chapter 9.5**.
- **Theory Meets Data, Chapters 1 and 2**.
- **Khan Academy's unit on Probability**.

## Probability theory

- Some things in life *seem* random.
    - e.g., flipping a coin or rolling a die 🎲.
- The **probability** of seeing "heads" when flipping a fair coin is $\frac{1}{2}$.
- One interpretation of probability says that if we flipped a coin infinitely many times, then $\frac{1}{2}$ of the outcomes would be heads.

Terminology




- **Experiment**: A process or action whose result is random.
    - e.g., rolling a die.
    - e.g., flipping a coin twice.
- **Outcome**: The result of an experiment.
    - e.g., the possible outcomes of rolling a six-sided die are 1, 2, 3, 4, 5, and 6.
    - e.g., the possible outcomes of flipping a coin twice are HH, HT, TH, and TT.
- **Event**: A set of outcomes.
    - e.g., the event that the die lands on a even number is the set of outcomes {2, 4, 6}.
    - e.g., the event that the die lands on a 5 is the set of outcomes {5}.
    - e.g., the event that there is at least 1 head in 2 flips is the set of outcomes {HH, HT, TH}.

## Terminology

- **Probability**: A number between 0 and 1 (equivalently, between 0% and 100%) that describes the likelihood of an event.
  - 0: The event never happens.
  - 1: The event always happens.
- Notation: If $A$ is an event, $P(A)$ is the probability of that event.

P(roll even)

1 2
3 4
5 6

event: even #

P(even #) = 3/6

## Equally-likely outcomes

- If all outcomes in event $A$ are equally likely, then the probability of $A$ is

$$P(A) = \frac{\text{\# of outcomes satisfying } A}{\text{total \# of outcomes}}$$

← fraction of outcomes in A

- **Example 1**: Suppose we flip a fair coin 3 times. What is the probability we see exactly 2 heads?

outcomes (8 = 2*2*2)

TTT HTT
TTH HTH
THT HHT
THH HHH

event

= 3/8

wrong way:
0H 2H
1H 3H

1/4

wrong because not all outcomes equally likely

Concept Check ✅ – Answer at **cc.dsc10.com**

I have three cards: red, blue, and green. What is the chance that I choose a card at random and it is green, then – **without putting it back** – I choose another card at random and it is red?

- A) $\frac{1}{9}$
- B) $\frac{1}{6}$
- C) $\frac{1}{3}$
- D) $\frac{2}{3}$
- E) None of the above.

green red, if you don't put green card back

$\frac{1}{3} * \frac{1}{2}$

outcomes – 6

without multiplicati

RB GR BG

RG GB BR

$\frac{1}{6}$

given: you know it happened

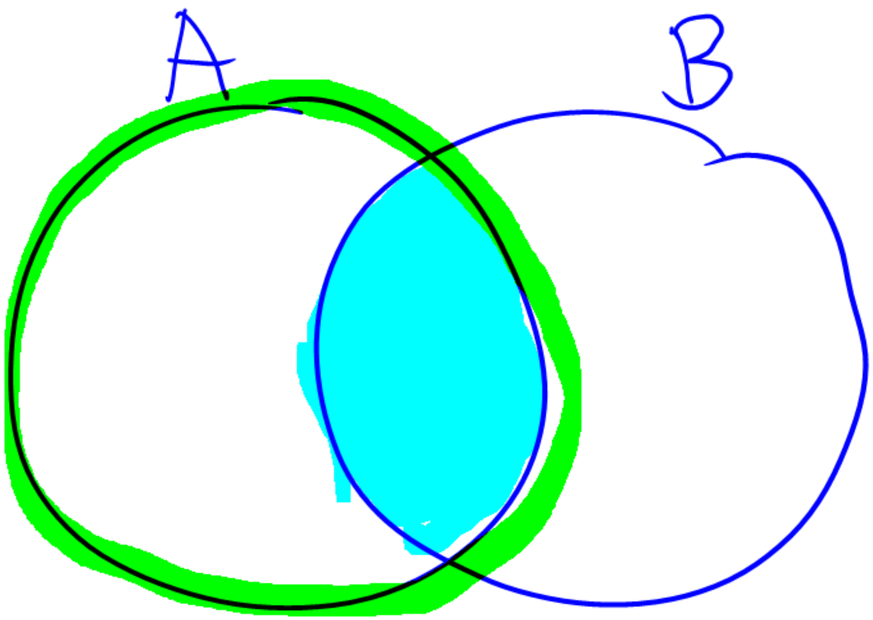
## Conditional probabilities

- Two events $A$ and $B$ can both happen. Suppose that we know $A$ has happened, but we don't know if $B$ has.
- If all outcomes are equally likely, then the conditional probability of $B$ given $A$ is:

$$P(B \text{ given } A) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\#\text{ of outcomes satisfying } A}$$

← fraction of A taken up by B

- Intuitively, this is similar to the definition of the regular probability of $B$, $P(B) = \frac{\#\text{ of outcomes satisfying } B}{\text{total } \#\text{ of outcomes}}$, if you restrict the set of possible outcomes to be just those in event $A$.

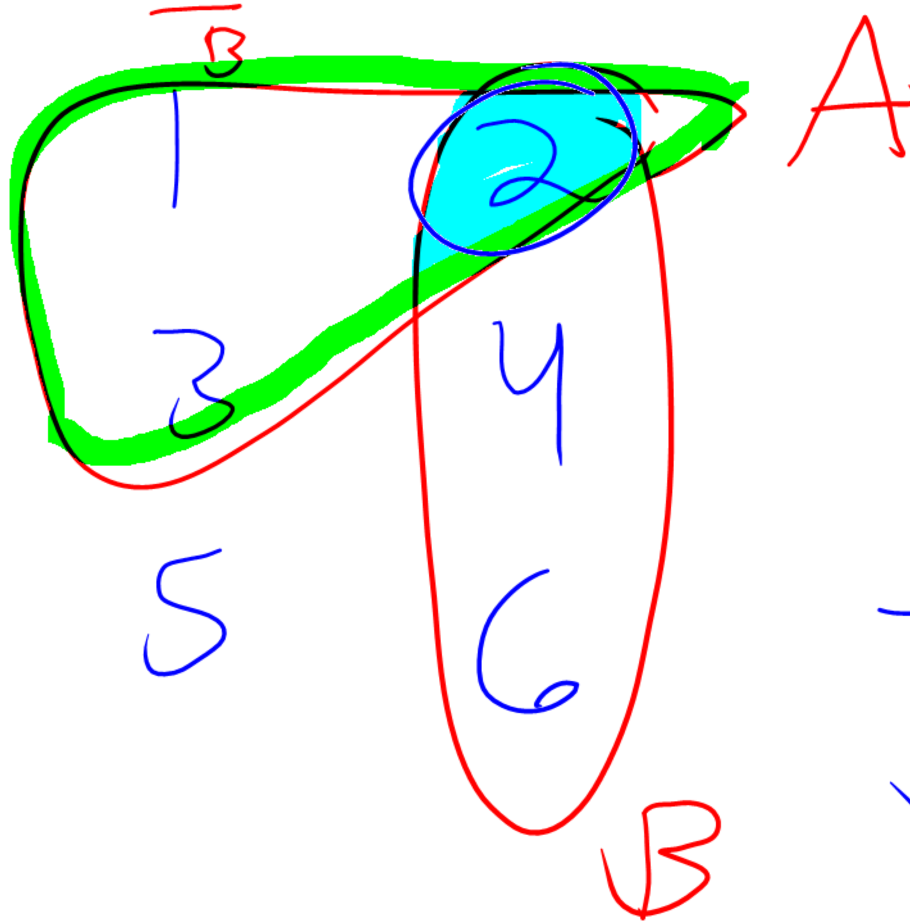


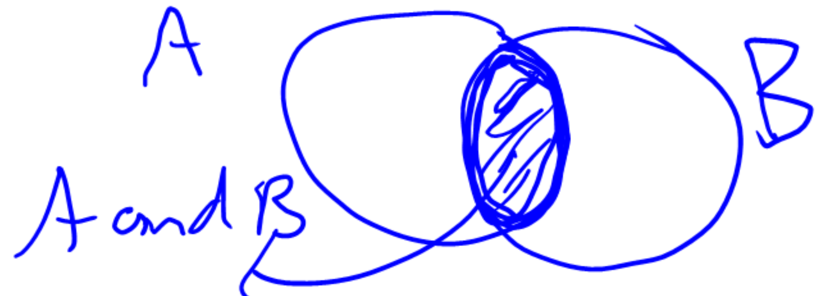
## Concept Check ✅ – Answer at **cc.dsc10.com**

$$P(B \text{ given } A) = \frac{\# \text{ of outcomes satisfying both } A \text{ and } B}{\# \text{ of outcomes satisfying } A}$$

I roll a six-sided die and don't tell you what the result is, but I tell you that it is 3 or less. What is the probability that the result is even?

- A) $\frac{1}{2}$
- B) $\frac{1}{3}$
- C) $\frac{1}{4}$
- D) None of the above.



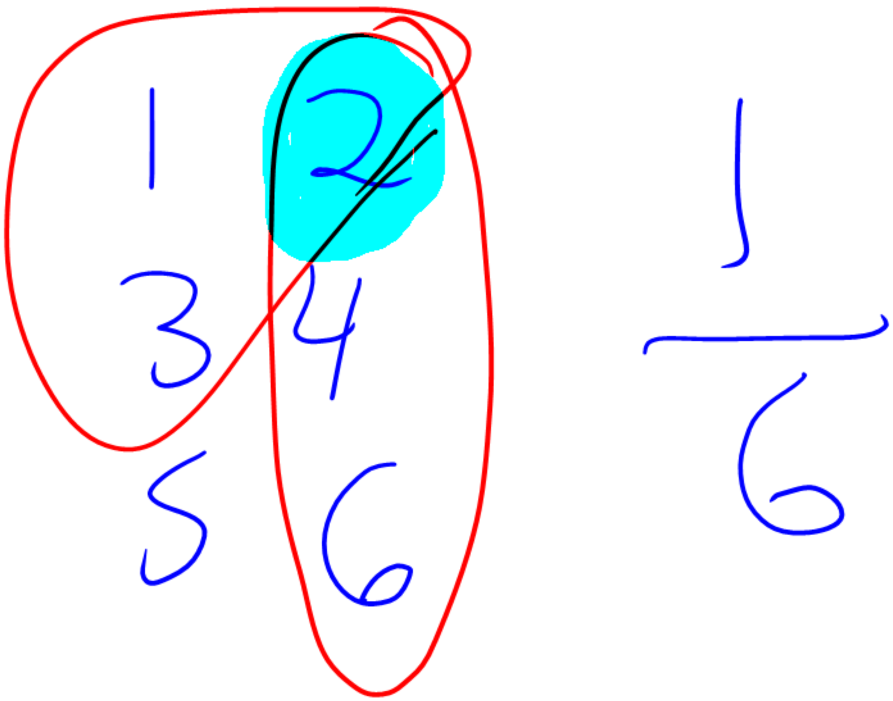



## Probability that two events both happen

- Suppose again that $A$ and $B$ are two events, and that all outcomes are equally likely. Then, the probability that both $A$ and $B$ occur is

$$P(A \text{ and } B) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\text{total } \#\text{ of outcomes}}$$

- **Example 2**: I roll a fair six-sided die. What is the probability that the roll is 3 or less **and** even?



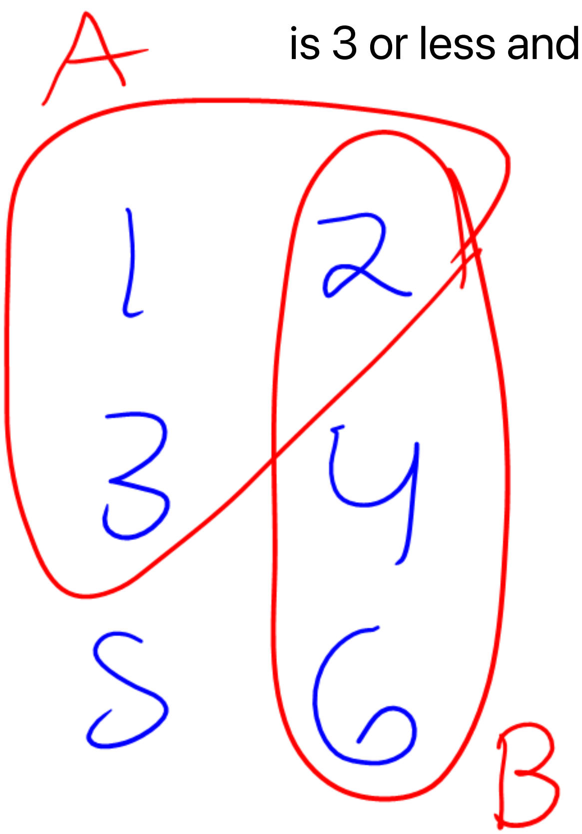
## The multiplication rule

- The multiplication rule specifies how to compute the probability of both $A$ and $B$ happening, even if all outcomes are not equally likely.

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

← general rule, always true

- **Example 2, again**: I roll a fair six-sided die. What is the probability that the roll is 3 or less and even?




$P(A) * P(B \text{ given } A)$

$= \frac{3}{6} * \frac{1}{3}$

$= \frac{1}{6}$

What if $A$ isn't affected by $B$? 🤔

- The multiplication rule states that, for any two events $A$ and $B$,

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

- What if knowing that $A$ happens doesn't tell you anything about the likelihood of $B$ happening?
    - Suppose we flip a fair coin three times.
    - The probability that the second flip is heads doesn't depend on the result of the first flip.
- Then, what is $P(A \text{ and } B)$?

when A and B don't impact each other, this simplifies to just P(B)

## Independent events

- Two events $A$ and $B$ are independent if $P(B \text{ given } A) = P(B)$, or equivalently if

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

→ special case of mult rule for independent events only

- **Example 3**: Suppose we have a coin that is **biased**, and flips heads with probability 0.7. Each flip is independent of all other flips. We flip it 5 times. What's the probability we see 5 heads in a row?

$P(\text{H on 1st and H on 2nd and} \ldots)$

$= P(\text{H on 1st}) * P(\text{H on 2nd}) * \ldots$

$= 0.7 * 0.7 * \ldots$

$= (0.7)^5$ not $0.7 * 5$

Probability that an event *doesn't* happen ← "complement rule"

- The probability that $A$ **doesn't** happen is $1 - P(A)$.
- For example, if the probability it is sunny tomorrow is 0.85, then the probability it is not sunny tomorrow is 0.15.

Prob(at least one Y) = 1 − P(all no's)

= 1 − 8/27 = $\frac{19}{27}$

Concept Check ✅ – Answer at **cc.dsc10.com**

Every time I call my grandma 👵, the probability that she answers her phone is $\frac{1}{3}$, independently for each call. If I call my grandma three times today, what is the chance that I will talk to her at least once?

at least one Y

- ~~A) $\frac{1}{3}$~~
- B) $\frac{2}{3}$
- C) $\frac{1}{2}$
- ~~D) 1~~ ← $\frac{1}{3} + \frac{1}{3} + \frac{1}{3}$
- E) None of the above.

outcomes – how many? 8, but not all equally likely

NNN → Prob(NNN) = $\frac{2}{3} * \frac{2}{3} * \frac{2}{3} = \frac{8}{27}$

YNY → Prob(YNY) = $\frac{1}{3} * \frac{2}{3} * \frac{1}{3} = \frac{2}{27}$

⋮

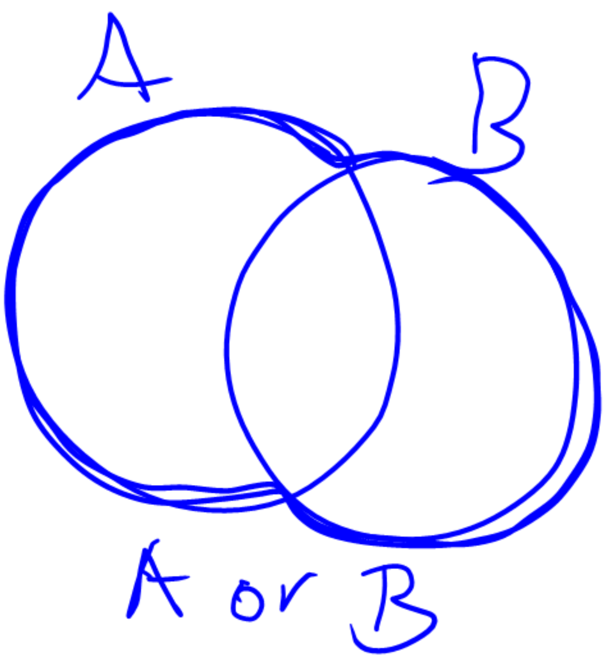
wrong way: $\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$ wrong because 6 is double counted (in A and B)

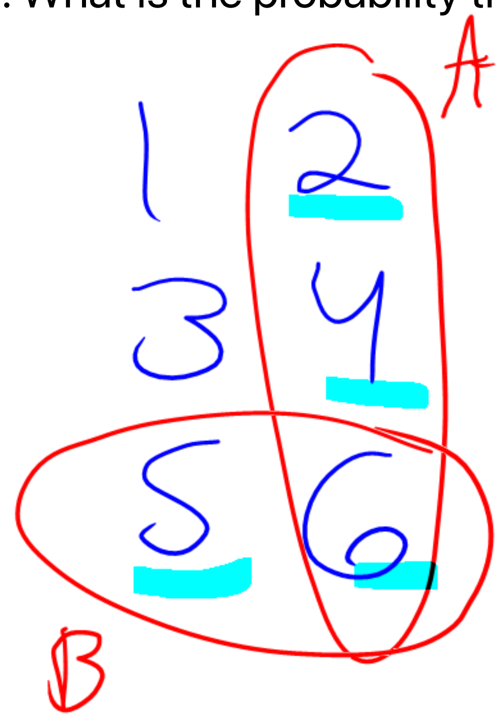
## Probability of either of two events happening

- Suppose again that $A$ and $B$ are two events, and that all outcomes are equally likely. Then, the probability that either $A$ or $B$ occur is

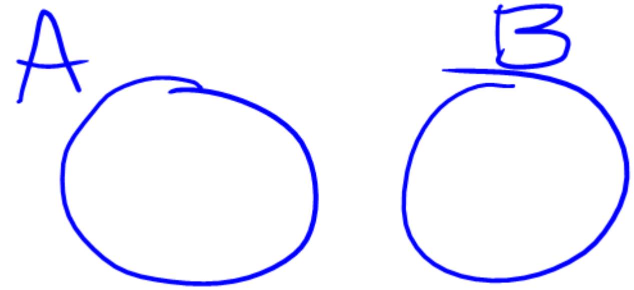
$$P(A \text{ or } B) = \frac{\# \text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \# \text{ of outcomes}}$$

- **Example 4**: I roll a fair six-sided die. What is the probability that the roll is even or at least 5?




$\frac{4}{6}$




## The addition rule

- Suppose that if $A$ happens, then $B$ doesn't, and if $B$ happens, then $A$ doesn't.
    - Such events are called **mutually exclusive** – they have **no overlap**.
- If $A$ and $B$ are any two mutually exclusive events, then

$$P(A \text{ or } B) = P(A) + P(B)$$

- **Example 5**: Suppose I have two biased coins, coin $A$ and coin $B$. Coin $A$ flips heads with probability 0.6, and coin $B$ flips heads with probability 0.3. I flip both coins once. What's the probability I see two different faces?

$$P(2 \text{ diff}) = P\left(\text{H on A and T on B} \text{ OR } \text{H on B and T on A}\right)$$

$$= P(\text{H on A and T on B}) + P(\text{H on B and T on A})$$

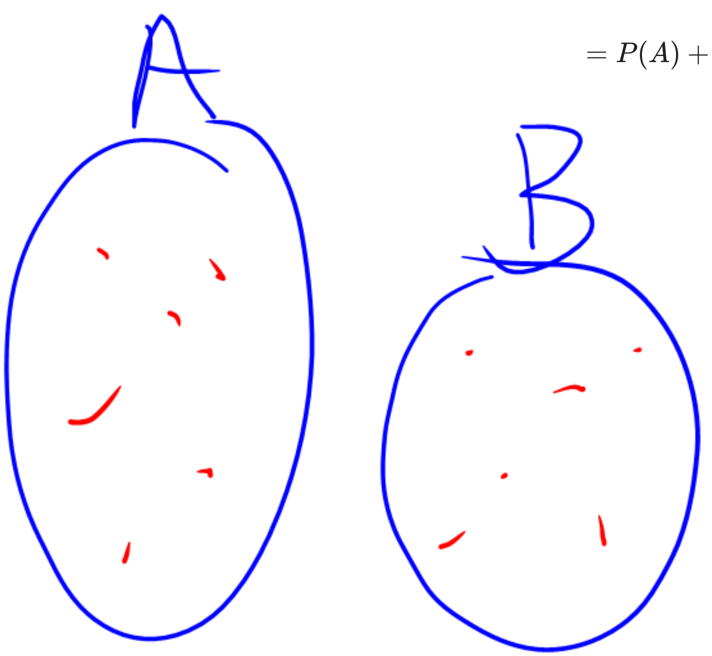
$$= \boxed{0.6 * 0.7 + 0.3 * 0.4}$$

## Aside: Proof of the addition rule for equally-likely events

You are not required to know how to "prove" anything in this course; you may just find this interesting.

If $A$ and $B$ are events consisting of equally likely outcomes, and furthermore $A$ and $B$ are mutually exclusive (meaning they have no overlap), then

$$
\begin{aligned}
P(A \text{ or } B) &= \frac{\#\text{ of outcomes satisfying either } A \text{ or } B}{\text{total }\#\text{ of outcomes}} \\
&= \frac{(\#\text{ of outcomes satisfying } A) + (\#\text{ of outcomes satisfying } B)}{\text{total }\#\text{ of outcomes}} \\
&= \frac{(\#\text{ of outcomes satisfying } A)}{\text{total }\#\text{ of outcomes}} + \frac{(\#\text{ of outcomes satisfying } B)}{\text{total }\#\text{ of outcomes}} \\
&= P(A) + P(B)
\end{aligned}
$$



## Summary, next time

- Probability describes the likelihood of an event occurring.
- There are several rules for computing probabilities. We looked at many special cases that involved equally-likely events.
- There are two general rules to be aware of:
  - The **multiplication rule**, which states that for any two events, $P(A \text{ and } B) = P(B \text{ given } A) \cdot P(A)$ .
  - The **addition rule**, which states that for any two **mutually exclusive** events, $P(A \text{ or } B) = P(A) + P(B)$.
- **Next time:** Simulations.