Lecture 11 – Probability

DSC 10, Winter 2024

Announcements

- Extra practice session is tonight. Problems are **here**.
    - This is the best way to prepare for the next quiz.
- Lab 3 is due **tomorrow at 11:59PM**.
- Quiz 3 is on **Monday in discussion**.
    - It covers lectures 8 through 11, which includes today.
- Homework 3 is due on **Thursday at 11:59PM**.

Agenda

We'll cover the basics of probability theory. This is a math lesson; take written notes ✍🏽.

Probability resources

Probability is a tricky subject. If it doesn't click during lecture or on the assignments, take a look at the following resources:

- **Computational and Inferential Thinking, Chapter 9.5**.
- **Theory Meets Data, Chapters 1 and 2**.
- **Khan Academy's unit on Probability**.

Probability theory

- Some things in life *seem* random.
    - e.g., flipping a coin or rolling a die 🎲.
- The **probability** of seeing "heads" when flipping a fair coin is $\frac{1}{2}$.
- One interpretation of probability says that if we flipped a coin infinitely many times, then $\frac{1}{2}$ of the outcomes would be heads.

ex. die

1 2
3 4
5 6

event: even #

Terminology

- **Experiment**: A process or action whose result is random.
  - e.g., rolling a die.
  - e.g., flipping a coin twice.

event: roll 5

- **Outcome**: The result of an experiment.
  - e.g., the possible outcomes of rolling a six-sided die are 1, 2, 3, 4, 5, and 6.
  - e.g., the possible outcomes of flipping a coin twice are HH, HT, TH, and TT.
- **Event**: A set of outcomes.
  - e.g., the event that the die lands on a even number is the set of outcomes {2, 4, 6}.
  - e.g., the event that the die lands on a 5 is the set of outcomes {5}.
  - e.g., the event that there is at least 1 head in 2 flips is the set of outcomes {HH, HT, TH}.

Terminology

- **Probability**: A number between 0 and 1 (equivalently, between 0% and 100%) that describes the likelihood of an event.
    - 0: The event never happens.
    - 1: The event always happens.
- Notation: If A is an event, $P(A)$ is the probability of that event.

P(roll even) or P(roll a 5)

Wrong way: 0H 1H $\boxed{2H}$ 3H → 1/4 (but not all equally likely!)

Equally-likely outcomes ← requirement

- If all outcomes in event A are equally likely, then the probability of A is

$$P(A) = \frac{\#\text{ of outcomes satisfying } A}{\text{total } \#\text{ of outcomes}}$$

← fraction of possible outcomes included in event

- **Example 1**: Suppose we flip a fair coin 3 times. What is the probability we see exactly 2 heads?

HHH THH
HHT THT
HTH TTH
HTT TTT

8 outcomes $(2*2*2)$

$P(2H) = \frac{3}{8}$

Concept Check ✅ – Answer at **cc.dsc10.com**

I have three cards: red, blue, and green. What is the chance that I choose a card at random and it is green, then – **without putting it back** – I choose another card at random and it is red?

- A) $\frac{1}{9}$
- B) $\frac{1}{6}$
- C) $\frac{1}{3}$
- D) $\frac{2}{3}$
- E) None of the above.

green red

$\frac{1}{3} * \frac{1}{2} = \frac{1}{6}$ ← works, using multiplication

P(G) * P(R given G)

without multiplication:

event

GR RG BG

GB RB BR

← equally likely

$P(GR) = \frac{1}{6}$

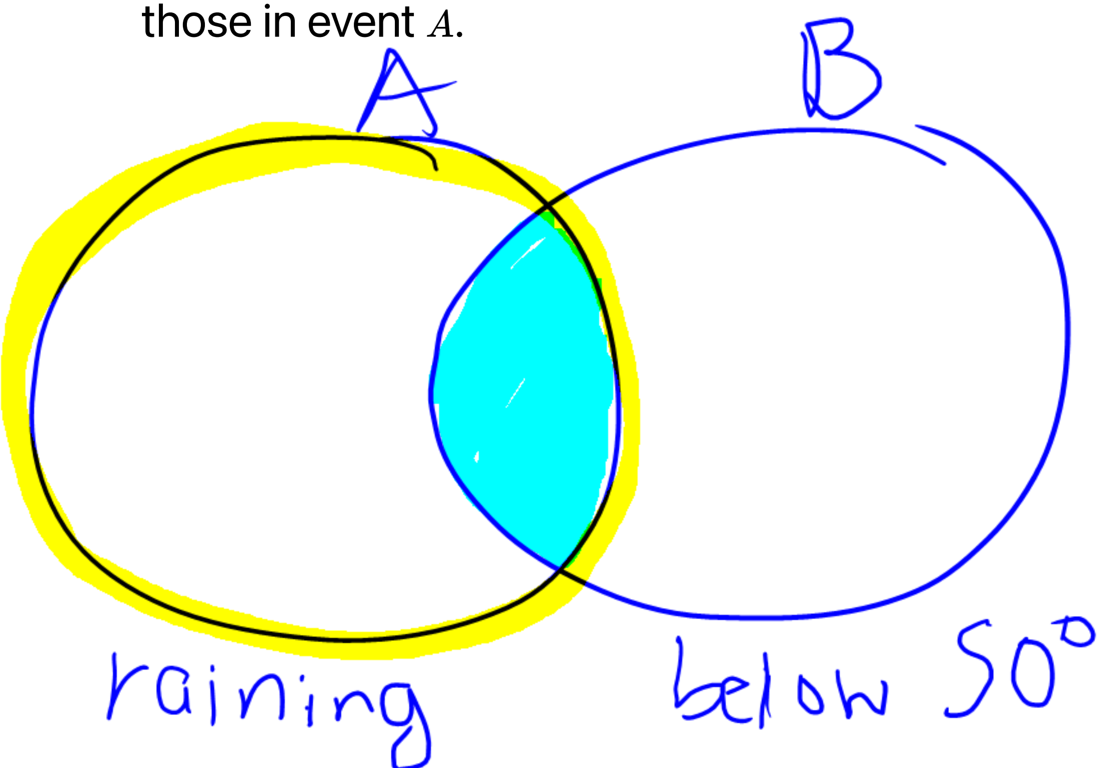
Conditional probabilities

- Two events A and B can both happen. Suppose that we know A has happened, but we don't know if B has.
- If all outcomes are equally likely, then the conditional probability of B given A is:

$$P(B \text{ given } A) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\#\text{ of outcomes satisfying } A}$$

= fraction of outcomes in A that are in B

- Intuitively, this is similar to the definition of the regular probability of B, $P(B) = \frac{\#\text{ of outcomes satisfying } B}{\text{total } \#\text{ of outcomes}}$, if you restrict the set of possible outcomes to be just those in event A.

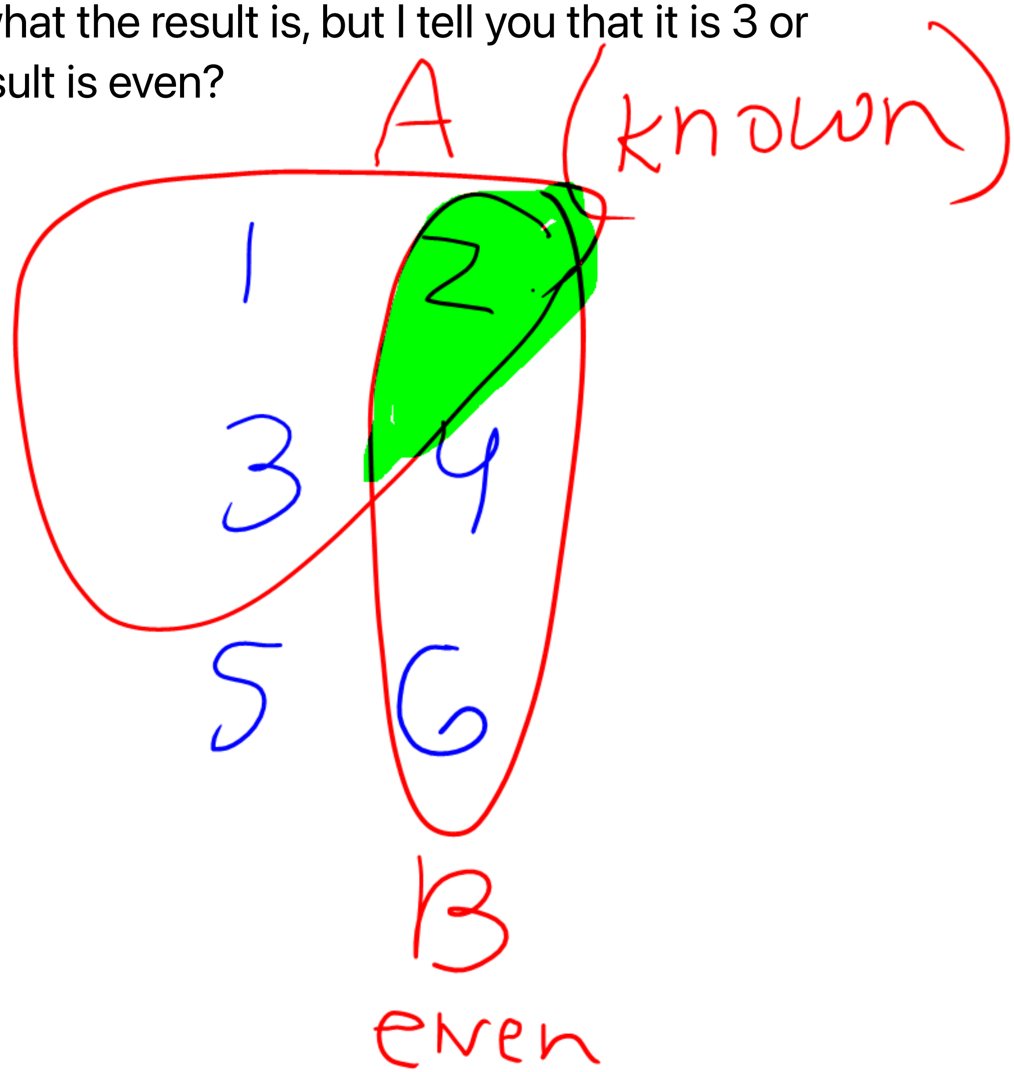


Concept Check ✅ – Answer at **cc.dsc10.com**

$$P(B \text{ given } A) = \frac{\# \text{ of outcomes satisfying both } A \text{ and } B}{\# \text{ of outcomes satisfying } A}$$

I roll a six-sided die and don't tell you what the result is, but I tell you that it is 3 or less. What is the probability that the result is even?

- A) $\frac{1}{2}$
- B) $\frac{1}{3}$
- C) $\frac{1}{4}$
- D) None of the above.



$$\boxed{\frac{1}{3}}$$

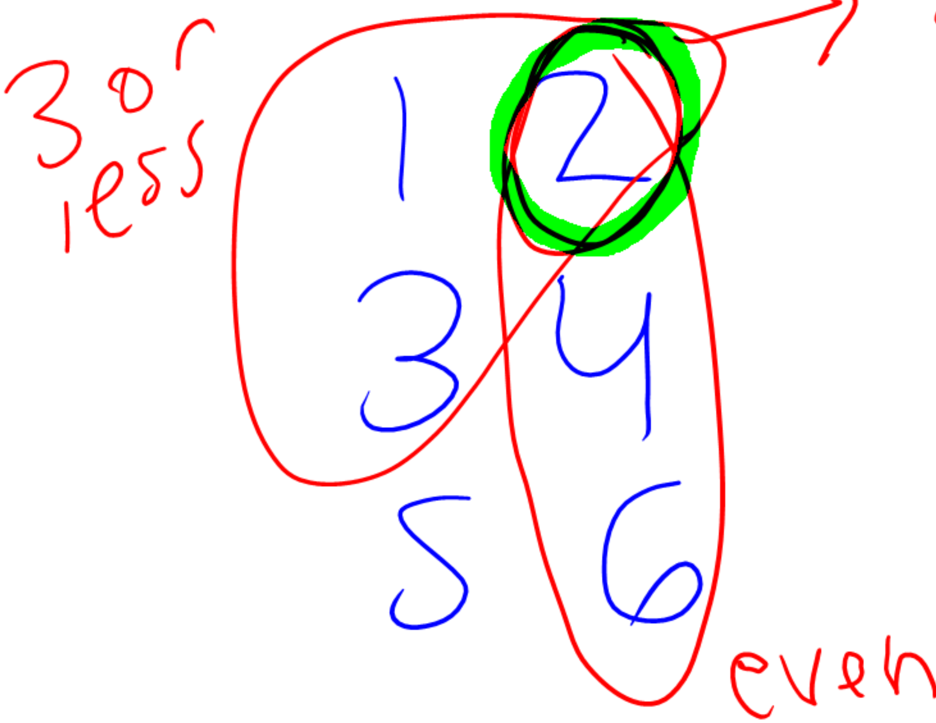
Probability that two events both happen

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that both A and B occur is

$$P(A \text{ and } B) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\text{total } \#\text{ of outcomes}}$$

- **Example 2**: I roll a fair six-sided die. What is the probability that the roll is 3 or less **and** even?

3 or less and even



$P(\text{3 or less and even}) = \frac{1}{6}$

The multiplication rule

- The multiplication rule specifies how to compute the probability of both A and B happening, even if all outcomes are not equally likely.

$$\boxed{P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)}$$

mult rule holds true always

- **Example 2, again**: I roll a fair six-sided die. What is the probability that the roll is 3 or less and even?

$$P(A) * P(B \text{ given } A)$$

$$\frac{3}{6} * \frac{1}{3} = \boxed{\frac{1}{6}}$$

What if A isn't affected by B? 🤔

- The multiplication rule states that, for any two events A and B,

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

- What if knowing that A happens doesn't tell you anything about the likelihood of B happening?
    - Suppose we flip a fair coin three times.
    - The probability that the second flip is heads doesn't depend on the result of the first flip.
- Then, what is $P(A \text{ and } B)$?

knowledge of A doesn't matter

Independent events

- Two events A and B are independent if $P(B \text{ given } A) = P(B)$, or equivalently if

$$\boxed{P(A \text{ and } B) = P(A) \cdot P(B)}$$

→ special case for when A, B independent

- **Example 3**: Suppose we have a coin that is **biased**, and flips heads with probability 0.7. Each flip is independent of all other flips. We flip it 5 times. What's the probability we see 5 heads in a row?

$P(1^{st}$ flip H and 2^{nd} flip H and ... 5^{th} flip H$)$

$= P(1^{st}$ flip H$) \cdot P(2^{nd}$ flip H$) \cdot \ldots \cdot P(5^{th}$ flip H$)$

$= 0.7 \cdot 0.7 \cdot \ldots \cdot 0.7$

$= (0.7)^5$ (not $0.7 * 5$)

Probability that an event *doesn't* happen

- The probability that A **doesn't** happen is $1 - P(A)$.
- For example, if the probability it is sunny tomorrow is 0.85, then the probability it is not sunny tomorrow is 0.15.

Concept Check ✅ – Answer at **cc.dsc10.com**

Every time I call my grandma 👵, the probability that she answers her phone is $\frac{1}{3}$, independently for each call. If I call my grandma three times today, what is the chance that I will talk to her at least once?

- ~~A) $\frac{1}{3}$~~ (prob. for one call, but you did 3)
- B) $\frac{2}{3}$
- C) $\frac{1}{2}$
- ~~D) 1~~ → $\frac{1}{3} + \frac{1}{3} + \frac{1}{3}$ (doesn't add)
- E) None of the above.

outcomes → not 7/8 because not all 8 outcomes equally likely

YYY – $\text{Prob}(YYY) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{27}$

YNY

7 outcomes include Y

NNN 1 outcome no Y

$P(NNN) = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{8}{27}$

how many outcomes? 8

$P(\text{at least one } Y) = 1 - \frac{8}{27} = \boxed{\frac{19}{27}}$

Probability of either of two events happening

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that either A or B occur is

$$\boxed{P(A \text{ or } B) = \frac{\# \text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \# \text{ of outcomes}}}$$

true whenever outcomes are equally likely

- **Example 4**: I roll a fair six-sided die. What is the probability that the roll is even or at least 5?

1 2
3 4
5 6

at least 5

even

$$\frac{4}{6}$$

(not ok to add $\frac{3}{6} + \frac{2}{6}$ because "6" gets counted twice)

1 2
3 4
5 6

even or 1

Prob is $\frac{4}{6} = \frac{3}{6} + \frac{1}{6}$

The addition rule

- Suppose that if A happens, then B doesn't, and if B happens, then A doesn't.
  - Such events are called **mutually exclusive** – they have **no overlap**.
- If A and B are any two mutually exclusive events, then

$$\boxed{P(A \text{ or } B) = P(A) + P(B)}$$

← Special case only when no overlap

- **Example 5**: Suppose I have two biased coins, coin A and coin B. Coin A flips heads with probability 0.6, and coin B flips heads with probability 0.3. I flip both coins once. What's the probability I see two different faces?

P(2 diff faces) = P((H for A and T for B) or (T for A and H for B))

= P(H for A and T for B) + P(T for A and H for B)

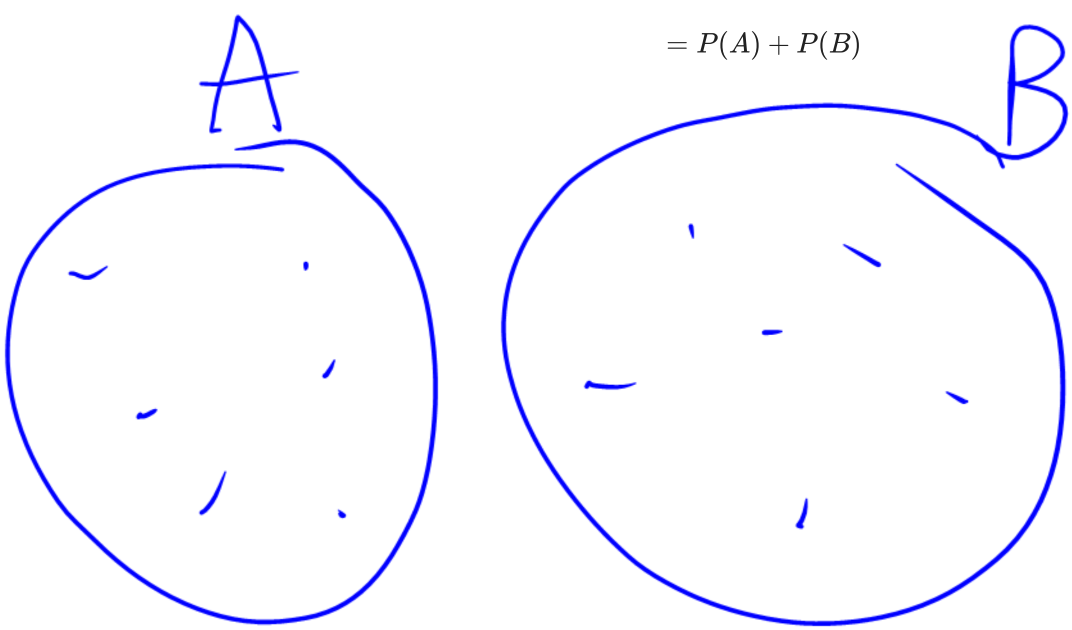
= P(H for A) * P(T for B) + P(T for A) * P(H for B)

$= \boxed{0.6 * 0.7 + 0.4 * 0.3}$

Aside: Proof of the addition rule for equally-likely events

You are not required to know how to "prove" anything in this course; you may just find this interesting.
If A and B are events consisting of equally likely outcomes, and furthermore A and B are mutually exclusive (meaning they have no overlap), then

$$
\begin{aligned}
P(A \text{ or } B) &= \frac{\#\text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \# \text{ of outcomes}} \\
&= \frac{(\#\text{ of outcomes satisfying } A) + (\#\text{ of outcomes satisfying } B)}{\text{total } \# \text{ of outcomes}} \\
&= \frac{(\#\text{ of outcomes satisfying } A)}{\text{total } \# \text{ of outcomes}} + \frac{(\#\text{ of outcomes satisfying } B)}{\text{total } \# \text{ of outcomes}} \\
&= P(A) + P(B)
\end{aligned}
$$



Summary, next time

- Probability describes the likelihood of an event occurring.
- There are several rules for computing probabilities. We looked at many special cases that involved equally-likely events.
- There are two general rules to be aware of:
  - The **multiplication rule**, which states that for any two events, $P(A \text{ and } B) = P(B \text{ given } A) \cdot P(A)$.
  - The **addition rule**, which states that for any two **mutually exclusive** events, $P(A \text{ or } B) = P(A) + P(B)$.
- **Next time:** Simulations.