Lecture 11 – Probability

DSC 10, Spring 2025

Agenda

We'll cover the basics of probability theory. This is a math lesson; take written notes ✍🏽.

Probability resources

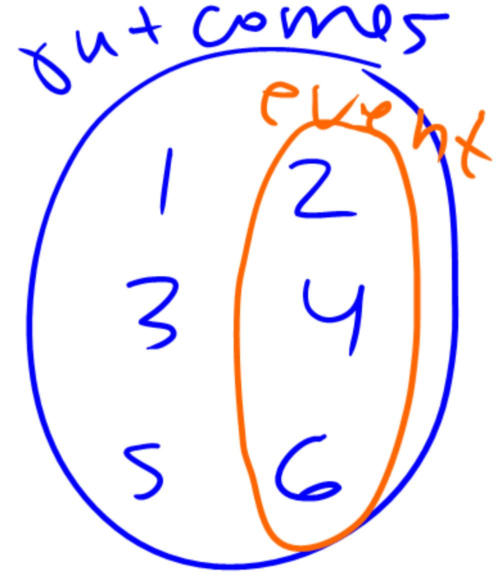
Probability is a tricky subject. If it doesn't click during lecture or on the assignments, take a look at the following resources:

- **Computational and Inferential Thinking, Chapter 9.5**.
- **Theory Meets Data, Chapters 1 and 2**.
- **Khan Academy's unit on Probability**.

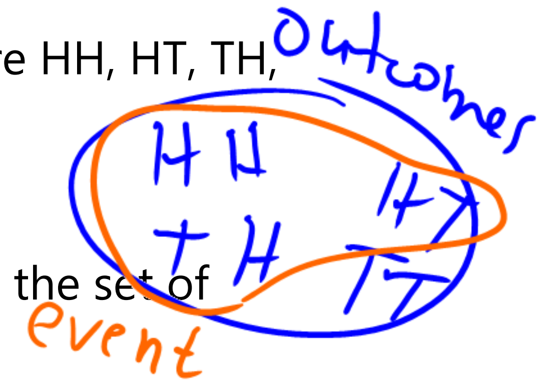
Probability theory

- Some things in life *seem* random.
    - e.g., flipping a coin or rolling a die 🎲.
- The **probability** of seeing "heads" when flipping a fair coin is $\frac{1}{2}$.
- One interpretation of probability says that if we flipped a coin infinitely many times, then $\frac{1}{2}$ of the outcomes would be heads.

Terminology



- **Experiment**: A process or action whose result is random.
  - e.g., rolling a die.
  - e.g., flipping a coin twice.
- **Outcome**: The result of an experiment.
  - e.g., the possible outcomes of rolling a six-sided die are 1, 2, 3, 4, 5, and 6.
  - e.g., the possible outcomes of flipping a coin twice are HH, HT, TH, and TT.



- **Event**: A set of outcomes.
  - e.g., the event that the die lands on a even number is the set of outcomes {2, 4, 6}.
  - e.g., the event that the die lands on a 5 is the set of outcomes {5}.
  - e.g., the event that there is at least 1 head in 2 flips is the set of outcomes {HH, HT, TH}.

Terminology

- **Probability**: A number between 0 and 1 (equivalently, between 0% and 100%) that describes the likelihood of an event.
    - 0: The event never happens.
    - 1: The event always happens.
- Notation: If A is an event, $P(A)$ is the probability of that event.

outcomes

event

1 2
3 4
5 6

P(even # when rolling die) = $\frac{3}{6} = \frac{1}{2}$

outcomes

HH HT
TH TT

event

P(at least one H when flipping 2 coins) = $\frac{3}{4}$

Equally-likely outcomes

- If all of the possible outcomes are equally likely, then the probability of A is

$$P(A) = \frac{\#\text{ of outcomes satisfying } A}{\text{total }\#\text{ of outcomes}} = \frac{\#\text{ good}}{\text{total }\#}$$

- **Example 1**: Suppose we flip a fair coin 3 times. What is the probability we see exactly 2 heads?

outcomes

event

HHH HHT
HTH HTT
THH THT
TTH TTT

$8 = 2 \cdot 2 \cdot 2$

$p = \frac{3}{8}$

Wrong

outcomes

0H 1H
2H 3H

1/8 3/8 3/8 1/8

$\frac{1}{4}$

R B G

Concept Check ✅ – Answer at cc.dsc10.com

event A

I have three cards: red, blue, and green. What is the chance that I choose a card at random and it is green, then – **without putting it back** – I choose another card at random and it is red?

event B

- A) $\frac{1}{9}$
- B) $\frac{1}{6}$
- C) $\frac{1}{3}$
- D) $\frac{2}{3}$
- E) None of the above.

$$\underbrace{\frac{1}{3}}_{G} * \underbrace{\frac{1}{2}}_{R} = \frac{1}{6}$$

outcomes

RG RB

GB GR

BG BR

$$P = \frac{1}{6}$$

$$P(A \text{ and } B)$$
$$= P(A) * P(B \text{ given } A)$$
$$= \frac{1}{3} * \frac{1}{2}$$
$$= \frac{1}{6}$$

Conditional probabilities

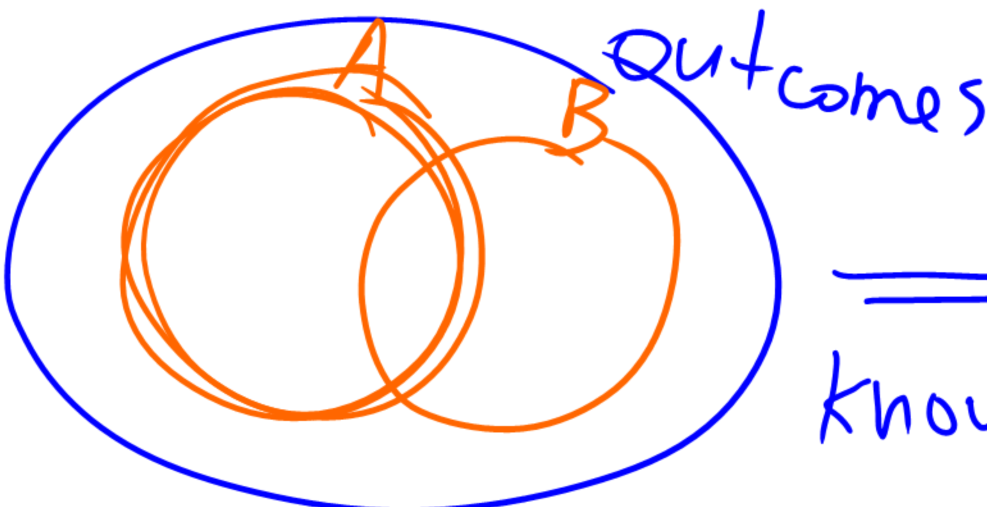
- Two events A and B can both happen. Suppose that we know A has happened, but we don't know if B has.
- If all outcomes are equally likely, then the conditional probability of B given A is:

$$P(B \text{ given } A) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\#\text{ of outcomes satisfying } A} = \frac{\text{good}}{\text{possible}}$$

- Intuitively, this is similar to the definition of the regular probability of B:

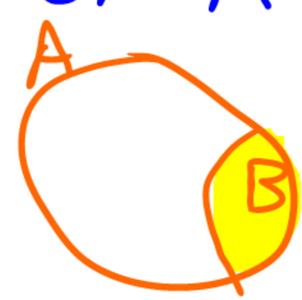
$$P(B) = \frac{\#\text{ of outcomes satisfying } B}{\text{total } \#\text{ of outcomes}}$$

if you restrict the set of possible outcomes to just those in event A.



$\Longrightarrow$ know A happened

zoom in on A



told / know A

Concept Check ✅ – Answer at **cc.dsc10.com**

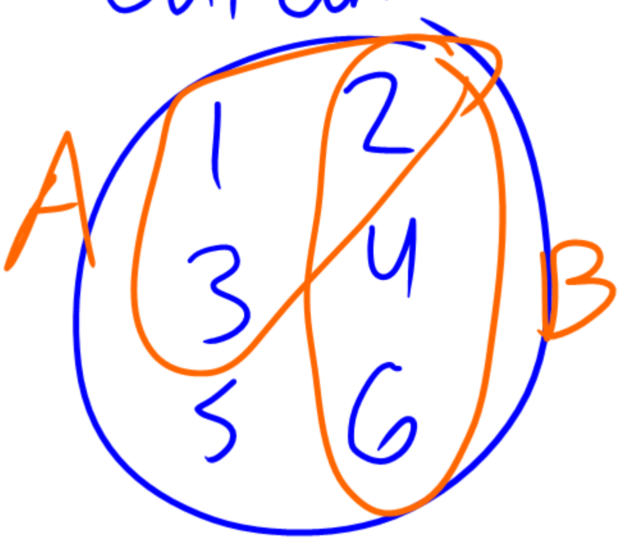
$$P(B \text{ given } A) = \frac{\# \text{ of outcomes satisfying both } A \text{ and } B}{\# \text{ of outcomes satisfying } A}$$

I roll a six-sided die and don't tell you what the result is, but I tell you that it is 3 or less. What is the probability that the result is even?

A B

- A) $\frac{1}{2}$
- B) $\frac{1}{3}$
- C) $\frac{1}{4}$
- D) None of the above.

Outcomes



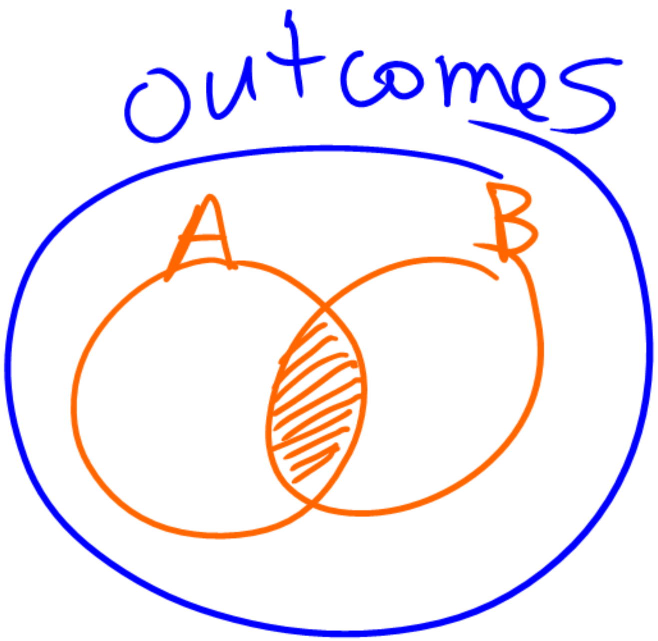
$\frac{1}{3}$

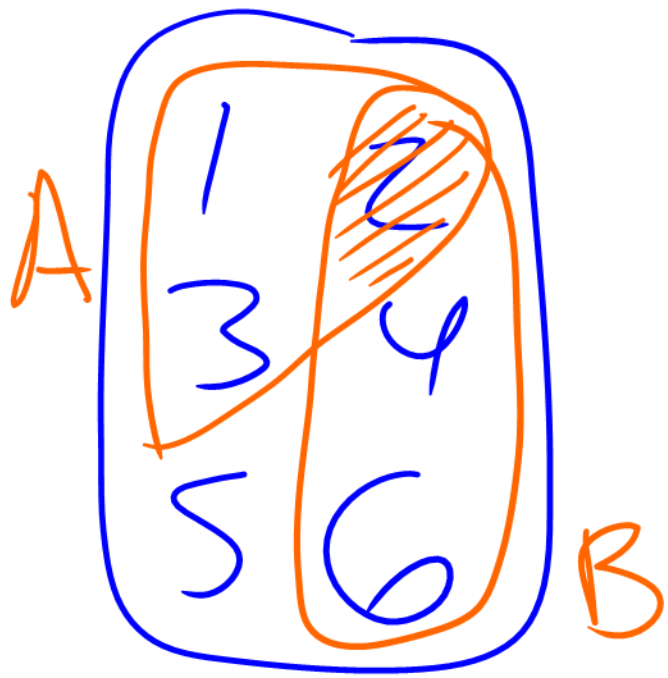
Probability that two events both happen

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that both A and B occur is

$$P(A \text{ and } B) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\text{total }\#\text{ of outcomes}}$$

- **Example 2**: I roll a fair six-sided die. What is the probability that the roll is 3 or less **and** even?





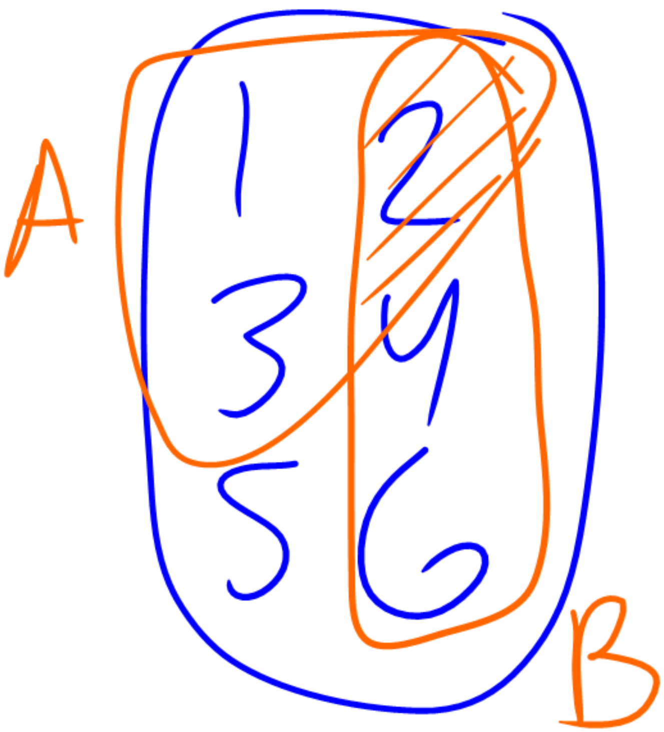
$$\frac{1}{6}$$

The multiplication rule

- The multiplication rule specifies how to compute the probability of both A and B happening, even if all outcomes are not equally likely.

$$\underline{P(A \text{ and } B)} = \underline{P(A)} \cdot \underline{P(B \text{ given } A)}$$

- **Example 2, again**: I roll a fair six-sided die. What is the probability that the roll is 3 or less and even?



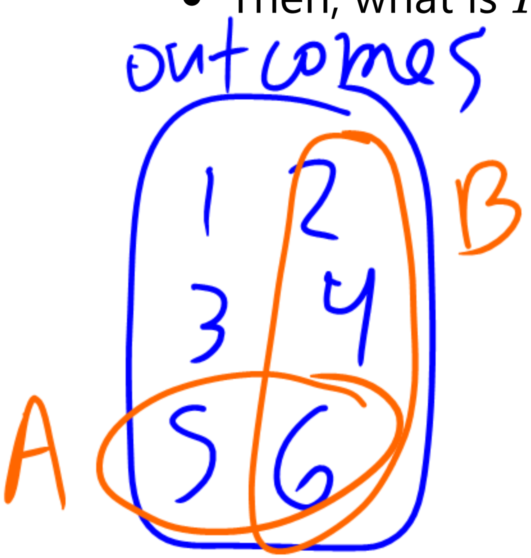
$$P(A \text{ and } B) = P(A) \times P(B \text{ given } A) = \frac{\cancel{3}}{6} \times \frac{1}{\cancel{3}} = \frac{1}{6}$$

What if A isn't affected by B? 🤔

- The multiplication rule states that, for any two events A and B,

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

- What if knowing that A happens doesn't tell you anything about the likelihood of B happening?
    - Suppose we flip a fair coin three times.
    - The probability that the second flip is heads doesn't depend on the result of the first flip.
- Then, what is $P(A \text{ and } B)$?



$$P(\text{even } \cancel{\text{given} \geq 5}) = \frac{1}{2}$$

$$P(\text{even}) = \frac{3}{6} = \frac{1}{2}$$

Independent events

- Two events A and B are independent if $P(B \text{ ~~given~~ } A) = P(B)$, or equivalently if

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

- **Example 3**: Suppose we have a coin that is **biased**, and flips heads with probability 0.7. Each flip is independent of all other flips. We flip it 5 times. What's the probability we see 5 heads in a row?

and

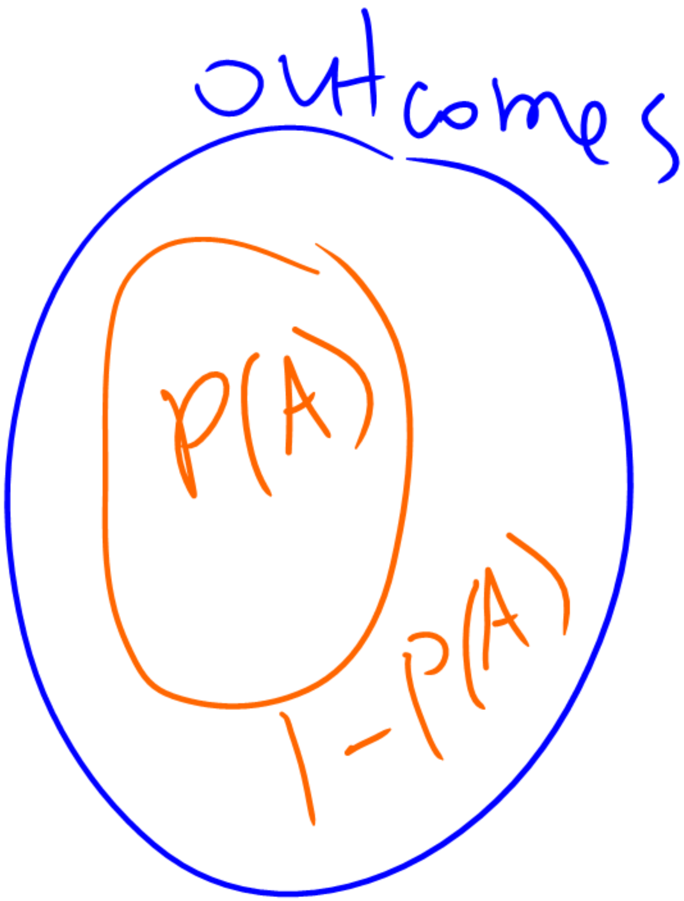
$$P(H\,H\,H\,H\,H)$$

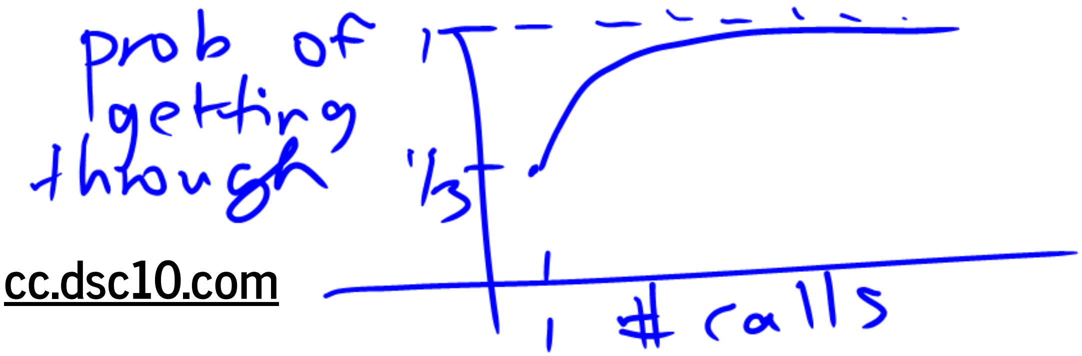
$$= P(H) * P(H) * \ldots * P(H)$$

$$= (0.7)^5$$

Probability that an event *doesn't* happen

- The probability that A **doesn't** happen is $1 - P(A)$.
- For example, if the probability it is sunny tomorrow is 0.85, then the probability it is not sunny tomorrow is 0.15.





Concept Check ✅ – Answer at cc.dsc10.com

Every time I call my grandma 👵, the probability that she answers her phone is $\frac{1}{3}$, independently for each call. If I call my grandma three times today, what is the chance that I will talk to her at least once?

- ~~A)~~ $\cancel{\frac{1}{3}}$
- B) $\frac{2}{3}$
- C) $\frac{1}{2}$
- ~~D) 1~~
- E) None of the above.

N = no

Y = yes

outcomes

NNY	YNY
NNN	YNN
NYN	YYN
NYY	YYY

$$P(NNY) = \frac{2}{3} * \frac{2}{3} * \frac{1}{3} = \frac{4}{27}$$

$$1 - P(NNN) = 1 - \frac{2}{3} * \frac{2}{3} * \frac{2}{3} = \frac{19}{27}$$

wrong: $P(\text{even}) = \frac{3}{6}$

$P(\geq 5) = \frac{2}{6}$

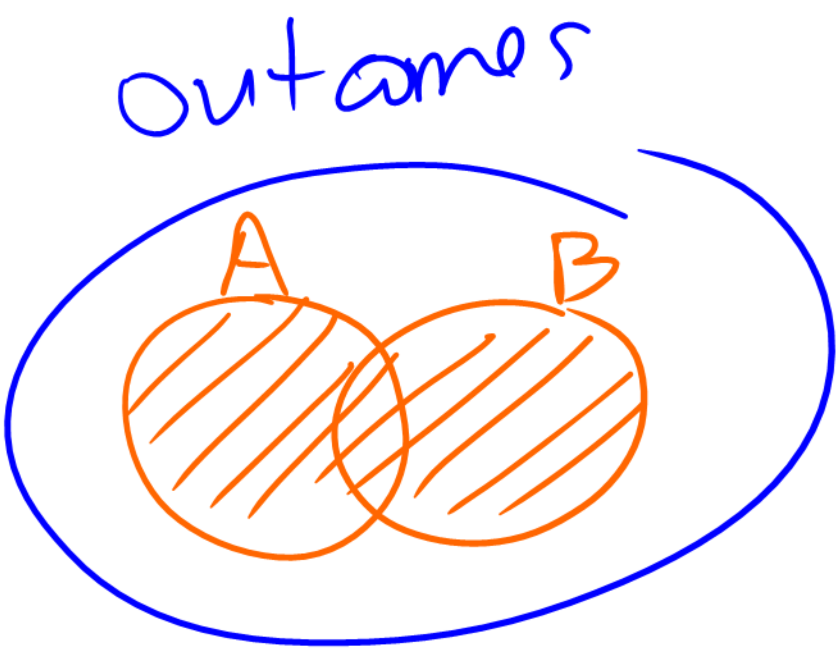
add $\frac{5}{6}$

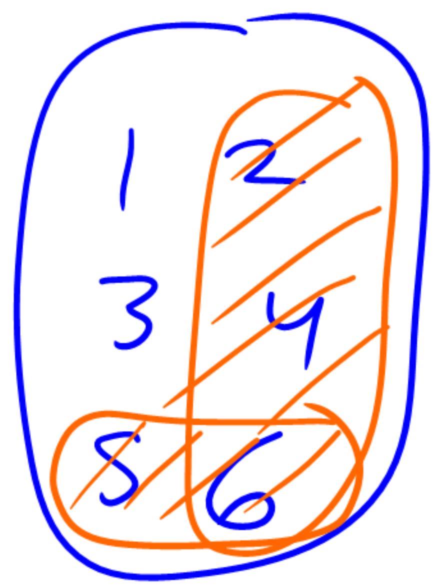
Probability of either of two events happening

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that either A or B occur is

$$P(A \text{ or } B) = \frac{\# \text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \# \text{ of outcomes}}$$

- **Example 4**: I roll a fair six-sided die. What is the probability that the roll is even or at least 5?





$\frac{4}{6}$

mult: and $\rightarrow *$ for ind. events

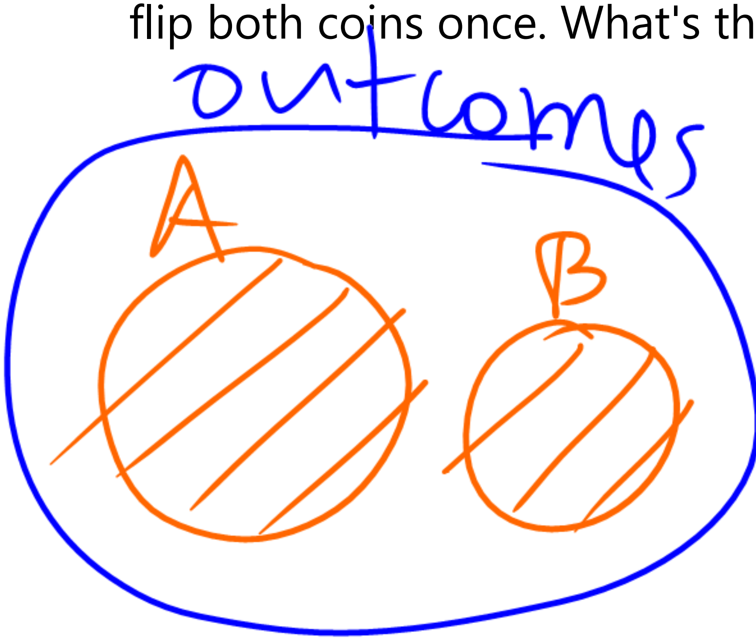
add: or $\Rightarrow +$ for mutually excl.

The addition rule

- Suppose that if A happens, then B doesn't, and if B happens, then A doesn't.
  - Such events are called **mutually exclusive** – they have **no overlap**.
- If A and B are any two mutually exclusive events, then

$$P(A \text{ or } B) = P(A) + P(B)$$

- **Example 5**: Suppose I have two biased coins, coin A and coin B. Coin A flips heads with probability 0.6, and coin B flips heads with probability 0.3. I flip both coins once. What's the probability I see two different faces?



outcomes

A	B
H	T
T	H
T	T
H	H

$P(H \text{ on } A \text{ and } T \text{ on } B) + P(H \text{ on } B \text{ and } T \text{ on } A)$

$0.6 * 0.7 + 0.3 * 0.4$

Aside: Proof of the addition rule for equally-likely events

You are not required to know how to "prove" anything in this course; you may just find this interesting.
If A and B are events consisting of equally likely outcomes, and furthermore A and B are mutually exclusive (meaning they have no overlap), then

$$
\begin{aligned}
P(A \text{ or } B) &= \frac{\#\text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \# \text{ of outcomes}} \\
&= \frac{(\#\text{ of outcomes satisfying } A) + (\#\text{ of outcomes satisfying } B)}{\text{total } \# \text{ of outcomes}} \\
&= \frac{(\#\text{ of outcomes satisfying } A)}{\text{total } \# \text{ of outcomes}} + \frac{(\#\text{ of outcomes satisfying } B)}{\text{total } \# \text{ of outcomes}} \\
&= P(A) + P(B)
\end{aligned}
$$

Summary, next time

- Probability describes the likelihood of an event occurring.
- There are several rules for computing probabilities. We looked at many special cases that involved equally-likely events.
- There are two general rules to be aware of:
    - The **multiplication rule**, which states that for any two events, $P(A \text{ and } B) = P(B \text{ given } A) \cdot P(A)$.
    - The **addition rule**, which states that for any two **mutually exclusive** events, $P(A \text{ or } B) = P(A) + P(B)$.
- **Next time:** Simulations.