

DSC 190

DATA STRUCTURES & ALGORITHMS

Today's Lecture

Last Time

- ▶ Time needed for BST operations is proportional to height.
- ▶ If tree is balanced, $h = \Theta(\log n)$
- ▶ If tree is unbalanced, $h = O(n)$

Today

- ▶ How do we ensure that tree is balanced?
- ▶ Approach 1: Complicated rules, red-black trees.
- ▶ Approach 2: Randomization
- ▶ We'll introduce **treaps**.

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Red-Black Trees

Self-Balancing BSTs

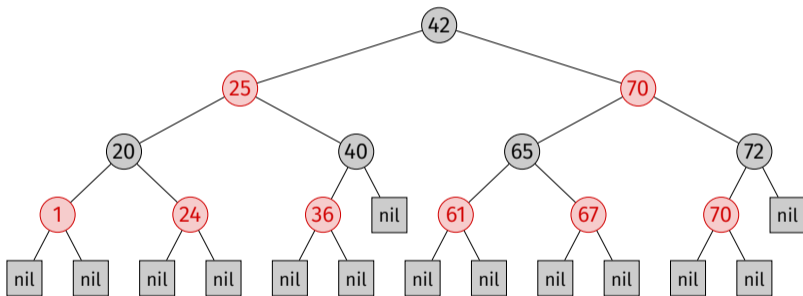
- ▶ We wish to ensure that the tree does not become unbalanced.
- ▶ Idea: If tree becoming unbalanced, it will balance itself.
- ▶ Several strategies, including **red-black** trees and **AVL** trees

Red-Black Trees

- ▶ A **red-black** tree is a BST whose nodes are colored **red** and **black**.
- ▶ Leaf nodes are “nil”.
- ▶ Must satisfy four additional properties:
 1. The root node is **black**.
 2. Every leaf node is **black**.
 3. If a node is **red**, both child nodes are **black**.
 4. For any node, all paths from the node to a leaf contain the same number of **black** nodes.

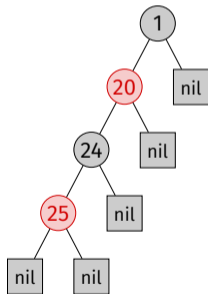


Example



Example

- ▶ This **not** a red-black tree.
 - ▶ Violates last property



Claim

If a red-black tree has n internal (non-nil) nodes, then the height is at most $2 \log(n + 1)$.

Proof Intuition¹

- ▶ All paths from root to a leaf are about the same length ($\approx h$).
- ▶ Therefore, the tree is close to balanced.
- ▶ So height is proportional to $\log n$

¹Formal proof proceeds by induction.

Non-Modifying Operations

- ▶ As a result, the non-modifying operations take $\Theta(\log n)$ time in red-black trees.
 - ▶ query
 - ▶ minimum/maximum
 - ▶ next smallest/largest
- ▶ Proof: these take $\Theta(h)$ time in any BST, and in a red-black tree $h = O(\log n)$.

Insertion and Deletion

- ▶ Standard BST `.insert` and `.delete` methods preserve BST, but **not** red-black properties.
- ▶ Insertion/deletion in a red-black tree is considerably more **complicated**.
- ▶ But both take $\Theta(\log n)$ time.

Implementing balanced trees is an exacting task and as a result balanced tree algorithms are rarely implemented except as part of a programming assignment in a data structures class².

Pugh, 1990

²For computer science majors.

Summary

- ▶ For red-black trees, worst cases:

query	$\Theta(\log n)$
minimum/maximum	$\Theta(\log n)$
next largest/smallest	$\Theta(\log n)$
insertion	$\Theta(\log n)$

- ▶ But they are **tricky** to implement.

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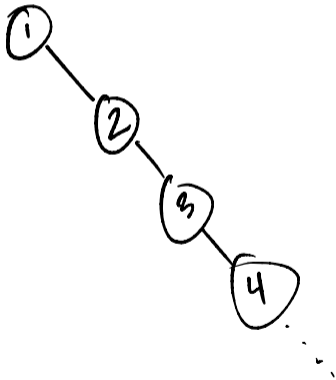
Randomization to the Rescue

Order Matters

- ▶ The structure of a BST depends on insertion order.

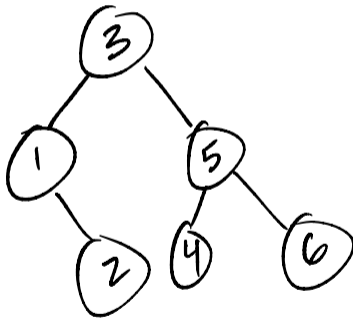
Example

- ▶ Insert 1,2,3,4,5,6 into BST, in that order.



Example

- ▶ Insert 3, 5, 1, 2, 4, 6 into BST, in that order.



Claim

The expected height of a BST built by inserting the keys in random order is $\Theta(\log n)$.

Idea

- ▶ To build a BST, take all n keys, shuffle them randomly, then insert.
- ▶ No need for Red-Black Trees, right?

Problem

- ▶ Usually don't have all the keys right now.
- ▶ This is a **dynamic set**, after all.
- ▶ The keys come to us in a stream, can't specify order.

Goal

- ▶ Design a data structure that **simulates** random insertion order without actually changing the order.

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Treaps

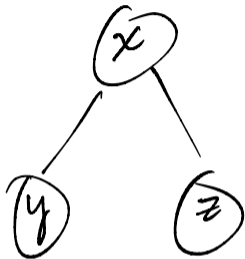
Randomization

- ▶ If insertions are in a random order, expected depth of a BST is $\Theta(\log n)$.
- ▶ But in **online** operation, we cannot randomize insertion order.
- ▶ Now: an elegant data structure simulating random insertion order in online operation.

First: Recall Heaps

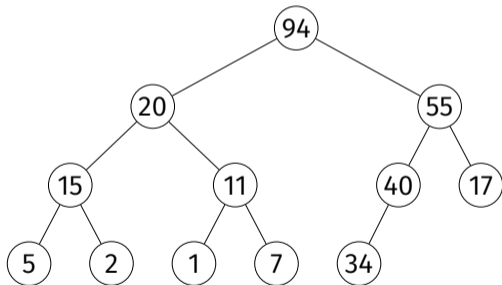
- ▶ A **max heap** is a **binary tree** where:
 - ▶ each node has a priority.
 - ▶ if y is a child of node x , then

$$y.\text{priority} \leq x.\text{priority}$$



Example

- ▶ This is a max heap:

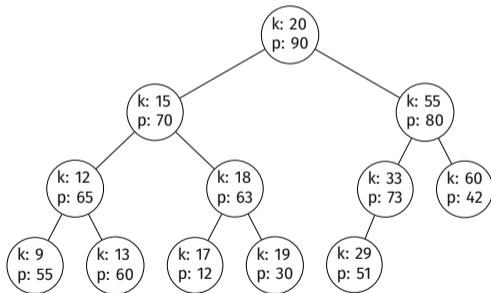


Treaps

- ▶ A **treap** is a binary tree in which each node has both a **key** and a **priority**.
- ▶ It is a **max heap** w.r.t. its priorities.
- ▶ It is a **binary search tree** w.r.t. its keys.

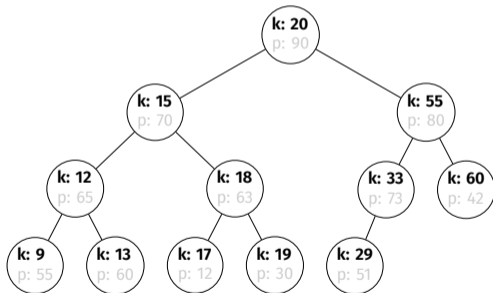
Example

- ▶ This is a treap:



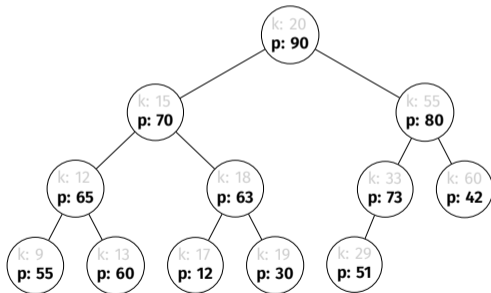
Example

- ▶ This is a treap:



Example

- ▶ This is a treap:



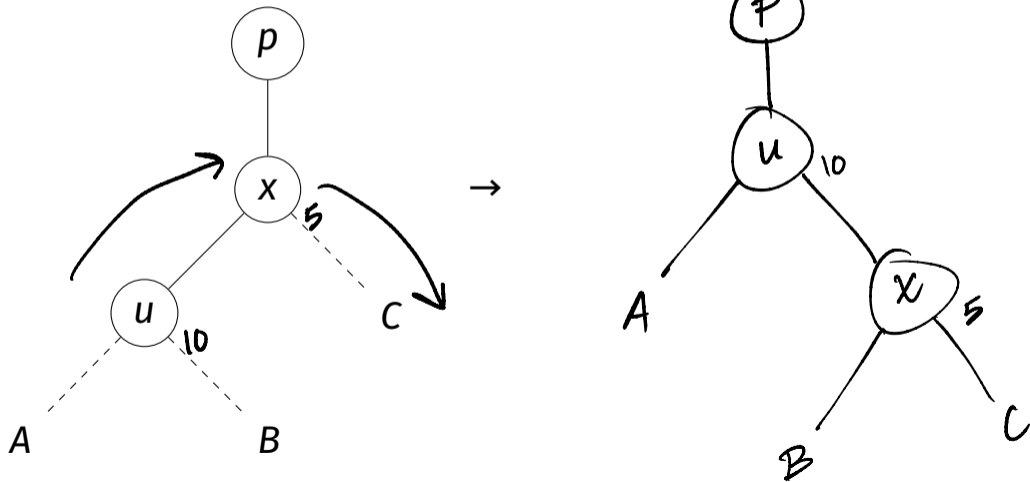
BST Operations

- ▶ Because a treap is a BST, querying, finding max/min by key is done the same.
- ▶ Insertion and deletion require care to preserve **heap** property.

Insertion

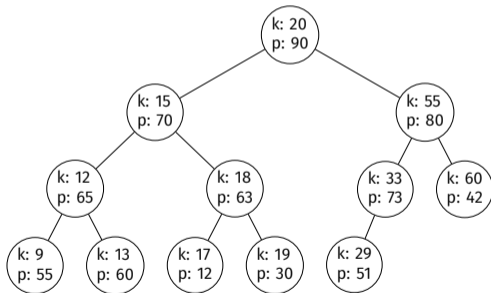
- ▶ Find place to insert node as usual.
- ▶ While priority of new node is $>$ than parent's:
 - ▶ Left rotate new node if it is the right child.
 - ▶ Right rotate new node if it is the left child.
- ▶ Rotate preserves BST, repeat until heap property satisfied.

(Right) Rotation



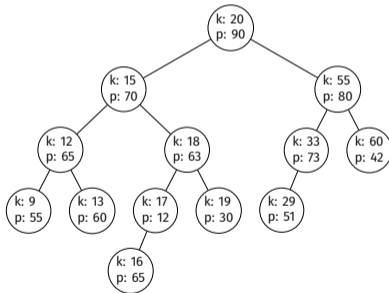
Example: Insertion

- ▶ Insert key: 16, priority: 65.



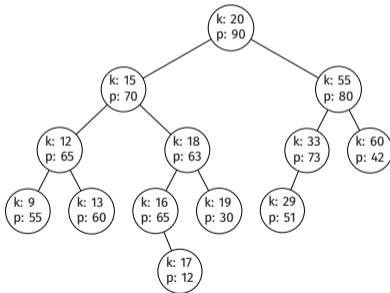
Example: Insertion

- ▶ Insert key: 16, priority: 65.



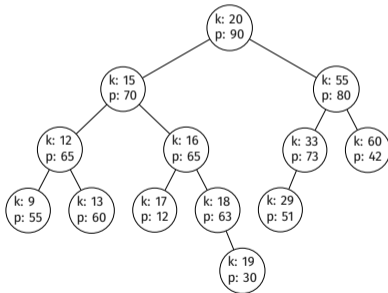
Example: Insertion

- ▶ Insert key: 16, priority: 65.



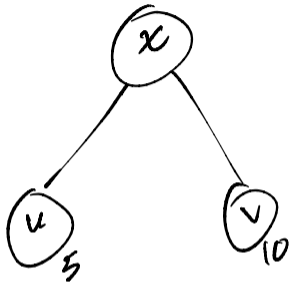
Example: Insertion

- ▶ Insert key: 16, priority: 65.



Deletion

- ▶ While node is not a leaf:
 - ▶ Rotate it with child of highest priority.
- ▶ Once it is a leaf, delete it.



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Treap Properties

Good Question

- ▶ Is it always possible to build a treap?

Claim

Given any set of (key, priority) pairs, inserting them one-by-one into a treap always results in a valid treap (no matter the insertion order).

Proof Idea

- ▶ Start with a treap (possibly empty).
- ▶ Inserting new (key, priority) preserves treap:
 - ▶ **BST**: rotation preserves BST property
 - ▶ **heap**: initially violated, but rotation repeated until it is satisfied

Claim

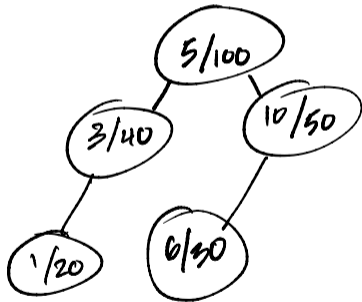
Given any set of (key, priority) pairs, if both keys and priorities are unique, then the treap is **unique**.

Claim

Corollary: Given any set of (key, priority) pairs, if both keys and priorities are unique, inserting them one-by-one into a treap results in the same treap, no matter the insertion order.

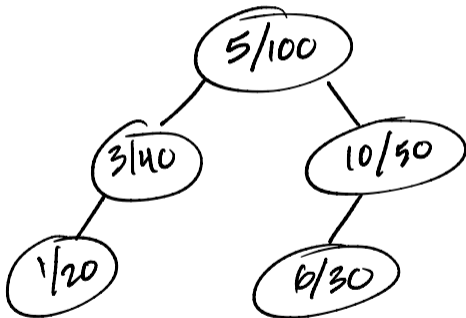
Example

- ▶ Insert $(3, 40)$, $(1, 20)$, $(10, 50)$, $(6, 30)$, $(5, 100)$, in that order

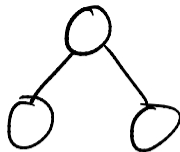


Example

- ▶ Insert (5, 100), (10, 50), (3, 40), (6, 30), (1, 20), in that order



Proof Idea



- ▶ Root node must be node w/ highest priority.
- ▶ Root's left (right) child must have highest priority among nodes with key $<$ ($>$) root key.
- ▶ Apply recursively.

Claim

Given any set of (key, priority) pairs, if both keys and priorities are unique, then inserting them one-by-one into a treap (in any order) results in the **same** BST one would obtain by inserting into a BST in decreasing order of priority.

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Randomized Binary Search Trees

Claim

Given any set of keys, if they are inserted into a BST in random order, the result is (almost surely) balanced. The expected height is $\Theta(\log n)$.

Claim

Given any set of (key, priority) pairs, if both keys and priorities are unique, then inserting them one-by-one into a treap (in any order) results in the **same** BST one would obtain by inserting into a BST in decreasing order of priority.

The Idea

- ▶ When inserting a node into a treap, generate priority **randomly**.
- ▶ The resulting treap will be the same tree as a BST built with nodes randomly ordered according to these priorities.
- ▶ It will almost surely be balanced.
- ▶ This is called a **randomized binary search tree**³.

³Sometimes people call these treaps

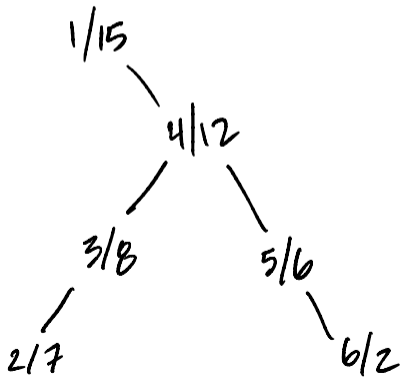
Warning

- ▶ Randomness does not mean that the result of, for example, a query has some probability of being incorrect.
- ▶ BST operations on treaps are always, 100% correct.
- ▶ The structure is random.

15 7 8 12 6 2

Example

- ▶ Insert 1, 2, 3, 4, 5, 6 into a treap, generating priorities randomly.



Time Complexities

- ▶ For randomized BSTs, expected times:

query	$\Theta(\log n)$
minimum/maximum	$\Theta(\log n)$
next largest/smallest	$\Theta(\log n)$
insertion	$\Theta(\log n)$

- ▶ Worst case times are $\Theta(n)$, but very rare

Comparison to Red-Black Trees

- ▶ When compared to red-black trees, randomized BSTs are:
 - ▶ same in terms of expected time;
 - ▶ perhaps slightly slower in practice;
 - ▶ **much** easier to implement/modify.

- ▶ Good trade-off for a data scientist!

Priority Hacks

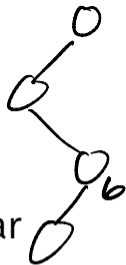
- ▶ Several interesting strategies for generating a new node's priority, beyond simply generating a random number.

Idea #1: Hashing

Justin

- ▶ Instead of randomly generating a number, hash the key to get priority.
- ▶ Works, provided hash function looks random.
- ▶ **Careful!** In python, `hash(300) == 300`

Idea #2: “Learning”



- ▶ Idea: Frequently-queried items should be near top of tree.
- ▶ When an item is queried, update its priority:
new priority = $\max(\text{old priority}, \text{random number})$

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Order Statistic Trees

Modifying BSTs

- ▶ More than most other data structures, BSTs must be modified to solve unique problems.
- ▶ Red-black trees are a pain to modify.
- ▶ Treaps/randomized BSTs are easy!

Order Statistics

- ▶ Given n numbers, the **k th order statistic** is the k th smallest number in the collection.

Example

[99, 42, -77, -12, 101]

- ▶ 1st order statistic: -77
- ▶ 2nd order statistic: -12
- ▶ 4th order statistic: 99

Exercise

Some special cases of order statistics go by different names. Can you think of some?

Special Cases

- ▶ **Minimum:** 1st order statistic.
- ▶ **Maximum:** n th order statistic.
- ▶ **Median:** $\lceil n/2 \rceil$ th order statistic⁴.
- ▶ **p th Percentile:** $\lceil \frac{p}{100} \cdot n \rceil$ th order statistic.

⁴What if n is even?

Computing Order Statistics

- ▶ Quickselect finds any order statistic in linear expected time.
- ▶ This is efficient for a static set.
- ▶ Inefficient if set is dynamic.

Goal

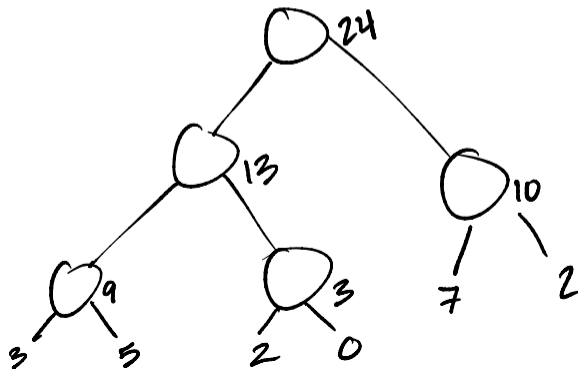
- ▶ Create a dynamic set data structure that supports fast computation of **any** order statistic.

Exercise

Does the “two heaps” trick from before work?

BST Solution

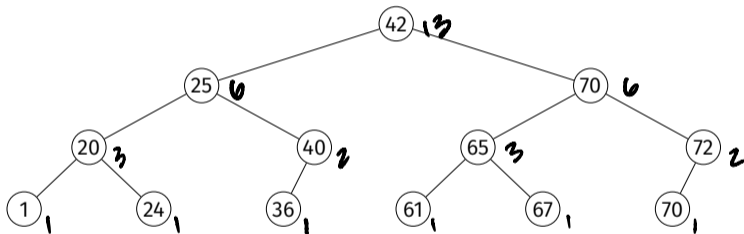
- ▶ For each node, keep attribute `.size`, containing # of nodes in subtree rooted at current node



$K=13$

Example: Insert/Delete

insert 30



Challenge

- ▶ ^{size} ~~number~~ It changes when nodes are inserted/deleted
- ▶ We must **modify** the code for insertion/deletion
- ▶ A pain with R-B tree; easy with treap!

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BSTs vs. Heaps

BSTs vs. Heaps

- ▶ Seemingly similar.
- ▶ Both are binary trees.
- ▶ Similar time complexities.

Summary

	Balanced BST	Binary Heap
get minimum/maximum	$\Theta(\log n)^5$	$\Theta(1)$
extract minimum/maximum	$\Theta(\log n)$	$\Theta(\log n)$
insertion	$\Theta(\log n)$	$\Theta(\log(n))$

⁵Can actually be optimized to $\Theta(1)$

Comparison

BSTs

- ▶ No cache locality
- ▶ Memory for pointers
- ▶ Maintains sorted order
- ▶ **Used for order statistics, queries**

Heaps

- ▶ Cache locality
- ▶ Use less memory
- ▶ Costly to query
- ▶ **Used for max/min**