DSC 190

Data Structures & Algorithms

Today's Lecture

Beyond Greedy

- Greedy algorithms are typically **fast**, but may not find the optimal answer.
- Brute force guarantees the optimal answer, but is **slow**.
- Can we guarantee the optimal answer and be faster than brute force?

Today

- The **backtracking** idea.
- It is a useful, general algorithm design technique[1].
- And the foundation of **dynamic programming**.

[1] Commonly seen in tech interviews

DSC 190

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The 0-1 Knapsack Problem

0-1 Knapsack

- Suppose you're a thief.
- You have a knapsack (bag) that can fit 100L.
- And a list of *n* things to possibly steal.

item	size (L)	price
TV	50	$400
iPad	2	$900
Printer	10	$100
⋮	⋮	⋮

- Goal: maximize total value of items you can fit in your knapsack.

Example

item	size (L)	price
1	50	$40
2	10	$25
3	80	$100
4	5	$10
5	20	$20
6	30	$6
7	8	$32
8	17	$34

In the bag: 1, 2

Total value: $40 + $25

Space remaining: 40

Greedy

- Does a greedy approach find the optimal?
- What do we mean by “greedy”?
- Idea #1: take most expensive available that will fit.

Example

item	size (L)	price
~~1~~	~~50~~	~~$40~~
2	10	$25
~~3~~	~~80~~	~~$100~~
4	5	$10
5	20	$20
~~6~~	~~30~~	~~$6~~
7	8	$32
~~8~~	~~17~~	~~$34~~

In the bag: 3, 8

Total value: $100 + $34 = $134

Space remaining: 3

Greedy, Idea #2

- We want items with high value for their size.
- Define “price density” = `item.price / item.size`
- Idea #2: take item with highest price density.

Example

item	size (L)	price	pd
~~1~~	~~50~~	~~$40~~	~~.8~~
~~2~~	~~10~~	~~$25~~	~~2.5~~
~~3~~	~~80~~	~~$100~~	~~1.25~~
~~4~~	~~5~~	~~$10~~	~~2~~
~~5~~	~~20~~	~~$20~~	~~1~~
~~6~~	~~30~~	~~$6~~	~~0.2~~
~~7~~	~~8~~	~~$32~~	~~4~~
~~8~~	~~17~~	~~$34~~	~~2~~

In the bag: 7, 2, 4, 8, 5, 6

Total value: $32 + $25 + $10 + $34 + $20 + $6 → 127

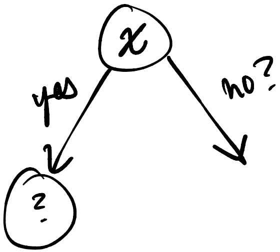
Space remaining: 10

Greedy is **Not Optimal**

- Claim: the best possible total value is \$157.
  - Items 2, 3, and 7.

Never Looking Back

- Once greedy makes a decision, it never looks back.
- This is why it may be suboptimal.
- **Backtracking**: go back to reconsider every previous decision.



DSC 190

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Backtracking

Backtracking

- Reconsider every decision.
- If we initially tried including x, also try *not* including x.

Backtracking

```python
def knapsack(items, bag_size):
    # choose item arbitrarily from those that fit in bag
    x = items.arbitrary_item(fitting_in=bag_size)

    # if None, it means there was no item that fit
    if x is None:
        return 0

    # assume x should be in bag, see what we get
    best_with = ...

    # backtrack: now assume x should not be in bag, see what we get
    best_without = ...

    return max(best_with, best_without)
```

Recursive Subproblems

- What is BEST(`items, bag_size`) if we assume that `x` **is** in the bag?
- Imagine choosing `x`.
  - Your current total value is `x.price`.
  - You have `bag_size - x.size` space left.
  - Items left to choose from: `items - x`.
- Clearly, you want the best outcome for *new* situation (subproblem).
- Answer: `x.price` + BEST(`items - x, bag_size - x.size`)

Recursive Subproblems

- What is BEST(`items, bag_size`) if we assume that `x` **is not** the bag?
- Imagine deciding `x` is not in the bag.
  - Your current total value is `0`.
  - You have `bag_size` space left.
  - Items left to choose from: `items - x`.
- Clearly, you want the best outcome for *new* situation (subproblem).
- Answer: `0` + BEST(`items - x, bag_size`)

Backtracking

```
def knapsack(items, bag_size):
    # choose item arbitrarily from those that fit in bag
    x = items.arbitrary_item(fitting_in=bag_size)

    # if None, it means there was no item that fit
    if x is None:
        return 0

    # assume x is in the bag, see what we get
    best_with = # knapsack(items - x, bag_size - x.size)

    # now assume x is not in bag, see what we get
    best_without = # knapsack(items - x, bag_size)

    return max(best_with, best_without)
```

x.price

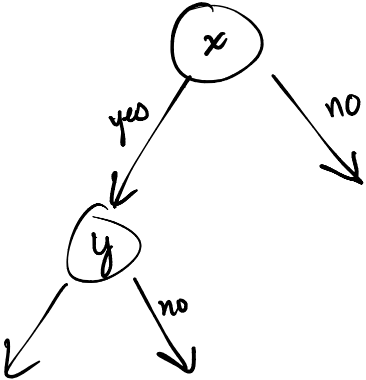
Backtracking

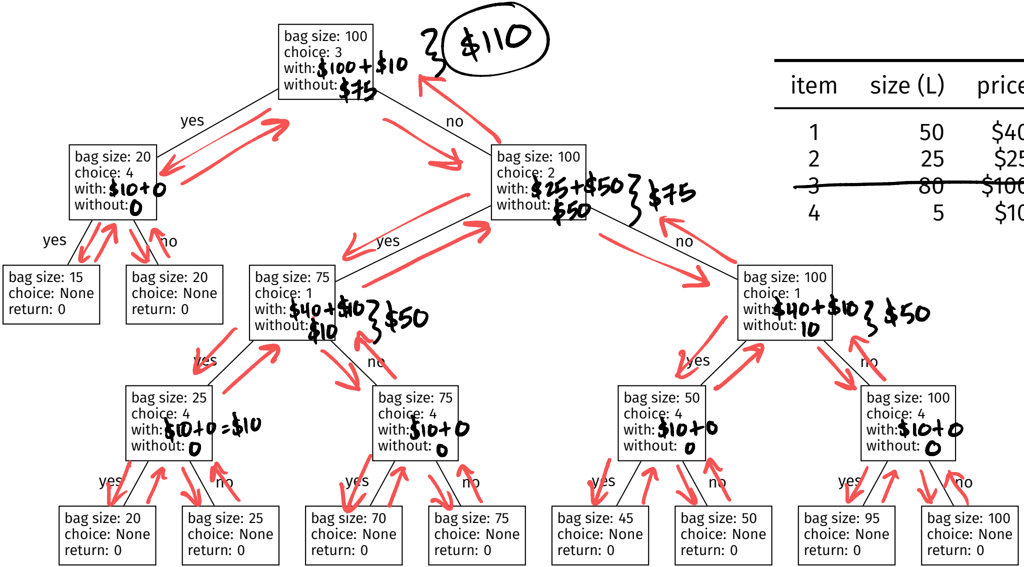
```python
def knapsack(items, bag_size):
    # choose item arbitrarily from those that fit in bag
    x = items.arbitrary_item(fitting_in=bag_size)

    # if None, it means there was no item that fit
    if x is None:
        return 0

    items.remove(x)
    best_with = knapsack(items, bag_size - x.size)
    best_without = knapsack(items, bag_size)
    items.replace(x)

    return max(best_with, best_without)
```

x.price +

Backtracking

- **Backtracking**: go back to reconsider every previous decision.
- Searches the whole tree.
- Can be thought of as a DFS on implicit tree.





item	size (L)	price
1	50	$40
2	25	$25
~~3~~	~~80~~	~~$100~~
4	5	$10

Exercise

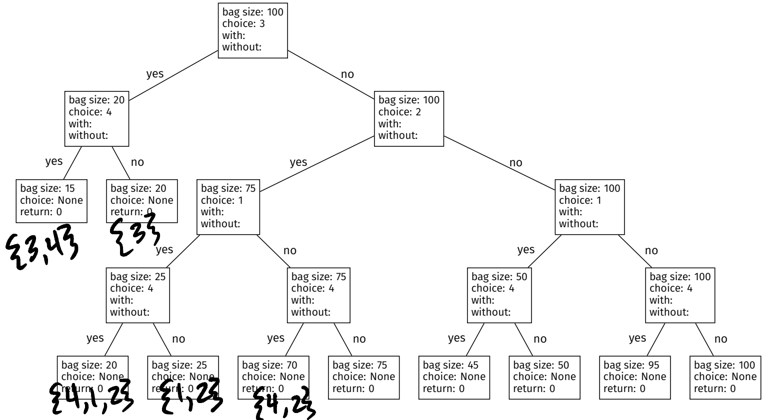
Is the backtracking solution guaranteed to find an optimal solution?

Yes!

- It tries every **valid** combination and keeps the best.
  - A combination of items is valid if they fit in the bag together.

Leaf Nodes

- Each leaf node is a different valid combination.



Exercise

Suppose instead of choosing an arbitrary node we choose most expensive. Does the answer change?

No!

- The choice of node is arbitrary.
- Call tree will change, but all valid combinations are tried.

Exercise

How does backtracking relate to the greedy approach? How would you change the code to make it greedy?

Summary

```
def knapsack_greedy(items, bag_size):
    # choose greedily
    x = items.most_valuable_item(fitting_in=bag_size)

    # if None, it means there was no item that fit
    if x is None:
        return 0

    # assume x is in the bag, see what we get
    items.remove(x)
    best_with = knapsack(items, bag_size - x.size)

    # in the greedy approach, we don't do this
    # best_without = # knapsack(items - x, bag_size)

    return best_with
```

x.price

DSC 190

DATA STRUCTURES & ALGORITHMS

Efficiency Analysis

A Benchmark

- Brute force: try every **possible** combination of items.
  - Even the **invalid** ones whose total size is too big.
  - Why? Hard to know which are invalid without trying them.

- There are $\Theta(2^n)$ possible combinations.

- So brute force takes $\Omega(2^n)$ time. **Exponential :(**

Time Complexity of Backtracking

```python
def knapsack(items, bag_size):
    # choose item arbitrarily from those that fit in bag
    x = items.arbitrary_item(fitting_in=bag_size)

    # if None, it means there was no item that fit
    if x is None:
        return 0

    items.remove(x)
    best_with = knapsack(items, bag_size - x.size)
    best_without = knapsack(items, bag_size)
    items.replace(x)

    return max(best_with, best_without)
```

x.price +

$$T(n) = T(n-1) + T(n-1) + \Theta(n)$$

$$T(n) = 2T(n-1) + \Theta(n)$$

Backtracking Takes Exponential Time

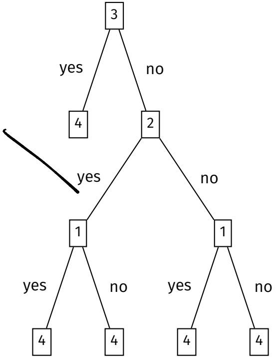
- ...in the worst case.
- This is just as bad as **brute force**.
- So why use it?
- Its worst case isn't always indicative of its practical performance.

Intuition

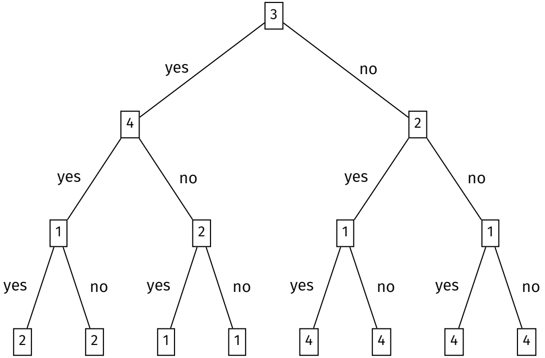
- Brute force tries all **possible** combinations.
- Backtracking tries all **valid** combinations.
- The number of valid combinations can be much less than the number of possible combinations.[2]

[2] Not always true!

Pruning



backtracking



brute force

Pruning

- Backtracking **prunes** branches that lead to invalid solutions.

Example

- 23 items with size/price chosen from Unif([23, ..., 46])
- Bag size is 46
- Brute force: 52 seconds.
- Backtracking: 4 milliseconds.

Example

- 300 items with size/price chosen from Unif([150, ..., 300])
- Bag size is 600
- Brute force: ? ($\approx 4.6 \times 10^{77}$ years)
- Backtracking: 30 seconds.

Backtracking Worst Case

- knapsack's **worst case** is when bag size is very large.
- All solutions are valid, aren't pruned.
- But this is actually an easy case!

```python
def knapsack_2(items, bag_size):
    if sum(item.size for item in items) <= bag_size:
        return sum(item.price for item in items)

    x = items.arbitrary_item(fitting_in=bag_size)

    if x is None:
        return 0

    items.remove(item)
    best_with = x.price + knapsack_2(items, bag_size - x.size)
    best_without = knapsack_2(items, bag_size)
    items.replace(x)

    return max(best_with, best_without)
```

Pruning

- This further prunes the tree, resulting in speedup for some inputs.

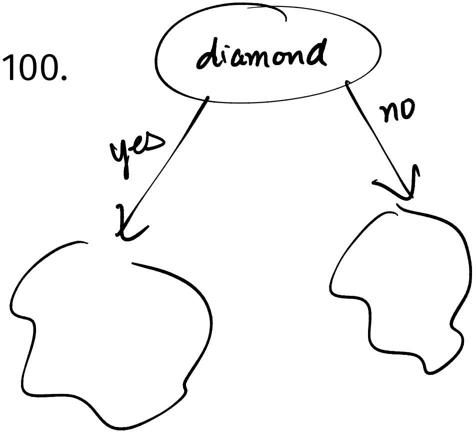
DSC 190

Data Structures & Algorithms

Branch and Bound

Example

- Suppose you have a bag of size 100.
- One of the items is a diamond.
  - Price: \$10,000. Size: 1
- The other 49 items are coal.
  - Price: \$1. Size: 1
- Do you even consider not taking the diamond?



Idea

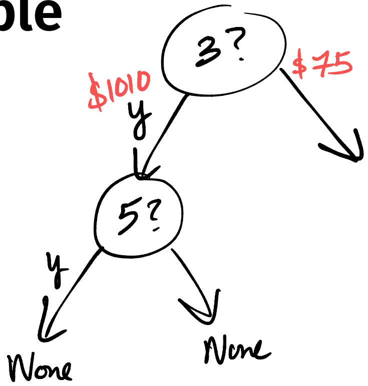
- Assume we take the diamond, compute best result.
- Find quick upper bound for not taking diamond.
- If upper bound is less than best for diamond, don't go down that branch.
- This is **branch and bound**; another way to prune tree.

Branch and Bound

```python
def knapsack_bb(items, bag_size, find_upper_bound):
    # try to make a good first choice
    x = items.item_with_highest_price_density(fitting_in=bag_size)

    if x is None:
        return 0

    items.remove(item)
    best_with = x.price + knapsack_bb(items, bag_size - x.size)

    if find_upper_bound(items, bag_size) < best_with:
        best_without = 0
    else:
        best_without = knapsack_bb(items, bag_size)

    items.replace(x)

    return max(best_with, best_without)
```

Example

item	size (L)	price
1	50	$40
2	25	$25
~~3~~	~~95~~	~~$1000~~
4	5	$10



Upper Bounds for Knapsack

- How do we get a good upper bound?
- One idea: the solution to the *fractional* knapsack problem upper bounds that for 0/1 knapsack.

DSC 190

Data Structures & Algorithms

Summary

Summary

- A backtracking approach is **guaranteed** to find an optimal answer.
- It is typically faster than brute force, but can still take **exponential time**.

Summary

- We can speed up backtracking by pruning:
- Three ways to prune:
  1. Prune invalid branches (default).
  2. Prune “easy” cases.
  3. Prune by branching and bounding.

Summary

- Next time: **dynamic programming**.
- We'll see it is just backtracking + memoization.