DSC 190 - Homework 06

Due: Wednesday, May 11

Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. Unless otherwise noted by the problem's instructions, show your work or provide some justification for your answer. Homeworks are due via Gradescope at 11:59 PM.

Problem 1.

In lecture, we designed a cost function for the embedding of n points into $\mathbb{R}^1$ using the coordinates of an embedding vector, $\vec{f}$:

$$\text{Cost}(\vec{f}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}(f_i - f_j)^2$$

We then said that this can also be written in the form:

$$\text{Cost}(\vec{f}) = \vec{f}^T L \vec{f},$$

where $L = D - W$ is the (unnormalized) graph Laplacian matrix.

Show that

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}(f_i - f_j)^2 = \vec{f}^T L \vec{f}.$$

in the simple setting where $n = 2$, and $\vec{f} = (f_1, f_2)^T$.

You may assume that the weight matrix, W, is symmetric.

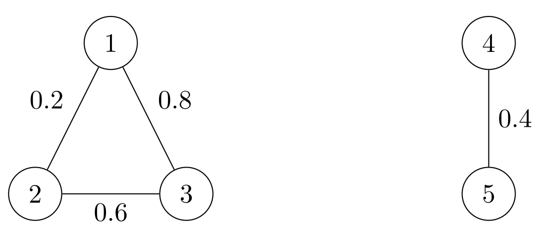
Problem 2.

In lecture, we saw that one minimizer of the cost function

$$\text{Cost}(\vec{f}) = \sum_{i}\sum_{j} w_{ij}(f_i - f_j)^2$$

is the vector $\vec{f} = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T$ which embeds all of the nodes of the graph to exactly the same number. The cost of this trivial embedding is zero, and we typically ignore it in favor of the eigenvector of the graph Laplacian with the *next* smallest positive eigenvalue for the embedding.

In the case where the graph has multiple connected components, however, there may be additional embeddings which have a cost of zero. For example, consider the similarity graph G shown below:



The weight of each edge is shown; the weight of non-existing edges is zero.

Find a normalized embedding vector $\vec{g}$ for the above graph such that $\text{Cost}(\vec{g}) = 0$ and $\vec{g} \perp \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T$; that is, $\vec{g}$ is orthogonal to the vector of all ones.