Lecture 12 – Foundations of Probability

DSC 40A, Fall 2021 @ UC San Diego

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Announcements

- **Please submit Survey 4!** → 5%
- Groupwork 5 due **tonight at 11:59pm**.
- Homework 5 due **Monday 11/8 at 11:59pm**.
- Homework 3 grades are out.

→ Look at the probability resources on the course website.

Agenda

- Complement, addition, and multiplication rules.
- Conditional probability.

Complement, addition, and multiplication rules

Sets

$\{1, 2, 3\} = \{3, 1, 2\}$

- A **set** is an unordered collection of items.

- A **sample space**, S, is the set of all possible outcomes of an experiment.

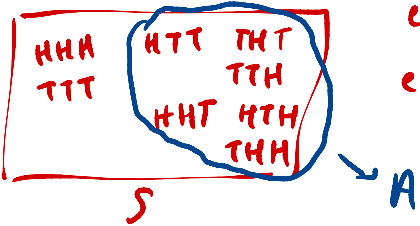
- An **event**, A is a subset of the sample space. In other words, an event is a set of outcomes.
  - Notationally: $A \subseteq S$.

- $|A|$ denotes the number of elements in set A.

$|A| = 6$

experiment: flip coin 3x

event A: flip 1 or 2 heads



Equally-likely outcomes

- If S is a sample space with n possible outcomes, and all outcomes are equally-likely, then the probability of any one outcome occurring is $\frac{1}{n}$.

- The probability of an event A, then, is

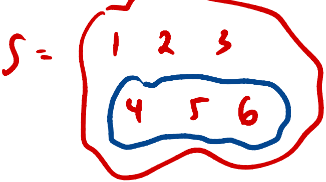
$$P(A) = \frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n} = \boxed{\frac{\text{\# of outcomes in A}}{\text{\# of outcomes in S}}} = \frac{|A|}{|S|}$$

- **Example:** Flipping a coin three times.

Complement rule

- Let A be an event with probability $P(A)$.
- Then, the event $\bar{A}$ is the **complement** of the event A. It contains the set of all outcomes in the sample space that are not in A.

e.g. die roll



$A = \{4, 5, 6\}$

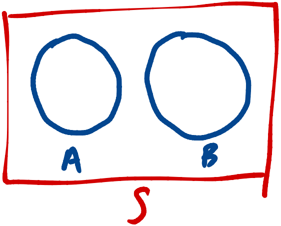
$\bar{A} = \{1, 2, 3\}$

- $P(\bar{A})$ is given by

$$P(\bar{A}) = 1 - P(A)$$

Addition rule

- We say two events are **mutually exclusive** if they have no overlap (i.e. they can't both happen at the same time).



e.g. rolling die

$A = \{1, 2\}$

$B = \{5, 6\}$

- If A and B are mutually exclusive, then the probability that A or B happens is

$$P(A \cup B) = P(A) + P(B)$$

A "or" B

Principle of inclusion-exclusion

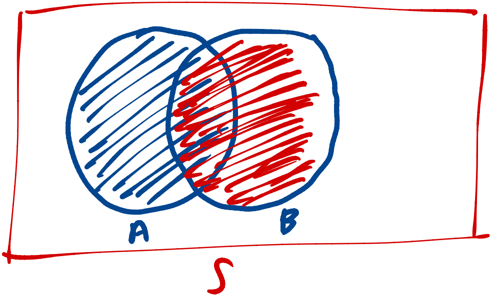
- If events A and B are not mutually exclusive, then the addition rule becomes more complicated.

e.g. rolling a die

$A = \{1, 2, 3, 4\}$

$B = \{4, 5, 6\}$

$A \cap B = \{4\}$



- In general, if A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

"union"

intersection

$P(A \text{ and } B)$

Discussion Question

Each day when you get home from school, there is a

- 0.3 chance your mom is at home
- 0.4 chance your brother is at home
- 0.25 chance that both your mom and brother are at home

When you get home from school today, what is the chance that neither your mom nor your brother are at home?

A) 0.3
B) 0.45
C) 0.55
D) 0.7
E) 0.75

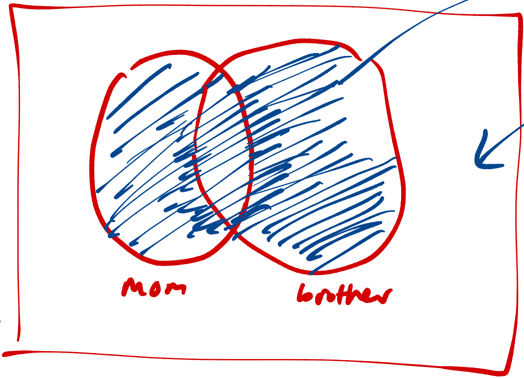
P(exactly 1 at home)

$= 0.3 + 0.4 - 0.25 - 0.25$

$= 0.2$

To answer, go to `menti.com` and enter 90 74 79.

0.45



$1 - 0.45$

$= \boxed{0.55}$

"at least 1" is at home

S

$$P(\text{mom or brother}) = P(\text{mom}) + P(\text{brother}) - P(\text{both})$$

$$= 0.3 + 0.4 - 0.25 = 0.45$$

Multiplication rule and independence

- The probability that events A and B both happen is

P(A and B) $\longrightarrow$ $$P(A \cap B) = P(A)P(B|A)$$

- $P(B|A)$ is read "the probability that B happens, given that A happened." It is a **conditional probability**.

- If $P(B|A) = P(B)$, events A and B are **independent**.
  - Intuitively, A and B are independent if knowing that A happened gives you no additional information about event B, and vice versa.

  - For two independent events,

$$P(A \cap B) = P(A)P(B)$$

Example: rolling a die

Let's consider rolling a fair 6-sided die. The results of each die roll are independent from one another.

- Suppose we roll the die once. What is the probability that the face is 1 and 2?

  0 → only see 1 number

- Suppose we roll the die once. What is the probability that the face is 1 or 2?

  $$\frac{2}{6} = \frac{1}{3}$$

  $$\underbrace{\boxed{\begin{matrix}1 & 2 & 3\\ 4 & 5 & 6\end{matrix}}}_{S}$$

Example: rolling a die

A : first roll not a 1
B : second roll not a 1
C : third roll not a 1

- Suppose we roll the die 3 times. What is the probability that the face 1 never appears in any of the rolls?

$$P(A \cap B \cap C) = P(A)\,P(B|A)\,P(C|A \cap B)$$

A, B, C independent

$$= P(A) \cdot P(B) \cdot P(C)$$

$$= \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = \left(\frac{5}{6}\right)^3$$

- Suppose we roll the die 3 times. What is the probability that the face 1 appears at least once?

$$1 - \left(\frac{5}{6}\right)^3 = 1 - \left(1 - \frac{1}{6}\right)^3$$

Example: rolling a die

$P(2 \text{ or } 4 \text{ or } 5 \text{ in single roll}) = \frac{1}{2}$

- Suppose we roll the die n times. What is the probability that only the faces 2, 4, and 5 appear?

$$\underbrace{\left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right)\cdots\left(\frac{1}{2}\right)}_{n \text{ times}} = \left(\frac{1}{2}\right)^n$$

$P(\text{second different from first}) = \frac{5}{6}$

- Suppose we roll the die twice. What is the probability that the two rolls have different faces?

	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

both same

$6^2 = 36$ outcomes

6 have doubles

$\rightarrow 36 - 6 = 30$ have diff faces

$\Rightarrow \frac{30}{36} = \frac{5}{6}$

Conditional probability

Conditional probability

- The probability of an event may **change** if we have additional information about outcomes.

- Starting with the multiplication rule, $P(A \cap B) = P(A)P(B|A)$, we have that

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

assuming that $P(A) > 0$.

$P(B \cap A) = P(B)P(A|B)$

Example: families

Suppose a family has two pets. Assume that it is equally likely that each pet is a dog or a cat. Consider the following two probabilities:

1. The probability that both pets are dogs given that **the oldest is a dog**.

2. The probability that both pets are dogs given that **at least one of them is a dog**.

Discussion Question

Are these two probabilities equal?

A) Yes, they're equal
B) No, they're not equal

To answer, go to `menti.com` **and enter 90 74 79.**

Example: families

Let's compute the probability that both pets are dogs given that **the oldest is a dog**.

$S = \{ cd, dc, cc, dd \}$

$A =$ oldest is a dog $= \{ cd, dd \}$

$B =$ both dogs $= \{ dd \}$ $\quad A \cap B = \{ dd \}$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

Example: families

Let's now compute the probability that both pets are dogs given that **at least one of them is a dog**.

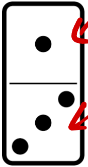
$S = \{cc, cd, dc, dd\}$

$A =$ at least 1 is dog $= \{cd, dc, dd\}$

$P(\text{both dogs} \mid \text{at least 1 is a dog}) = \frac{1}{3}$

Example: dominoes (source: 538)

In a set of dominoes, each tile has two sides with a number of dots on each side: zero, one, two, three, four, five, or six. There are 28 total tiles, with each number of dots appearing alongside each other number (including itself) on a single tile.



one tile

(0,0), (0,1), ..., (0,6)
(1,1), (1,2)... (1,6)
(2,2)....

write out all possibilities if you want

0 : 7 options for other side

\+

1 : 6 options for other side {1, 2, 3, 4, 5, 6}

\+

2 : 5

\+

⋮

6 : 1

$= 1+2+\dots+7$

$= \boxed{28}$

Example: dominoes (source: 538)

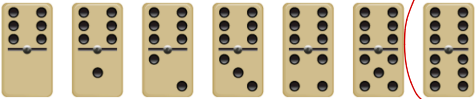
Question 1: What is the probability of drawing a "double" from a set of dominoes — that is, a tile with the same number on both sides?

7 doubles: {00, 11, 22, 33, 44, 55, 66}

$$P(\text{double}) = \frac{7}{28} = \boxed{\frac{1}{4}}$$

Example: dominoes (source: 538)

Question 2: Now your friend picks a random tile from the set and tells you that at least one of the sides is a 6. What is the probability that your friend's tile is a double, with 6 on both sides?

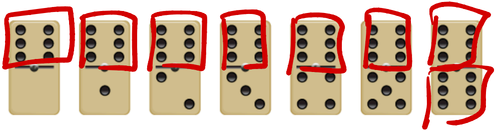


$\frac{1}{7}$

$$P(\text{double} \mid \text{at least 1 6}) = \frac{P(\text{double and at least 1 6})}{P(\text{at least 1 6})} = \frac{\frac{1}{28}}{\frac{7}{28}} = \boxed{\frac{1}{7}}$$

Example: dominoes (source: 538)

Question 3: Now you pick a random tile from the set and uncover only one side, revealing that it has six dots. What is the probability that this tile is a double, with six on both sides?



56 "halves"

$$\frac{\frac{2}{56}}{\frac{8}{56}}$$

You only looked at 1/2 a tile!!!

$\Rightarrow$ 8 halves have a 6

$\Rightarrow$ 2 halves are part of a double

$\rightarrow \frac{2}{8} = \boxed{\frac{1}{4}}$

$$P(\text{both halves are } 6 \mid \text{one half is a } 6)$$

$$= \frac{P(\text{both halves are } 6 \text{ and one half is } 6)}{P(\text{one half is a } 6)}$$

$$= \frac{\frac{2}{56}}{\frac{8}{56}} = \frac{2}{8} = \frac{1}{4}$$

Simpson's Paradox (source: nih.gov)

	Treatment A	Treatment B
Small kidney stones	81 successes / 87 (93%)	234 successes / 270 (87%)
Large kidney stones	192 successes / 263 (73%)	55 successes / 80 (69%)
Combined	273 successes / 350 (78%)	289 successes / 350 (83%)

Discussion Question

Which treatment is better?

A) Treatment A for all cases.
B) Treatment B for all cases.
C) Treatment A for small stones and B for large stones.
D) Treatment A for large stones and B for small stones.

To answer, go to `menti.com` and enter 90 74 79.

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Simpson’s Paradox occurs when an association between two variables exists when the data is divided into subgroups, but reverses or disappears when the groups are combined.

- See more in DSC 80.

Summary, next time

Summary

- $\bar{A}$ is the complement of event A. $P(\bar{A}) = 1 - P(A)$.
- Two events A, B are mutually exclusive if they share no outcomes, i.e. they don't overlap. In this case, the probability that A happens or B happens is $P(A \cup B) = P(A) + P(B)$.
- More generally, for any two events, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- The probability that events A and B both happen is $P(A \cap B) = P(A)P(B|A)$.
  - $P(B|A)$ is the probability that B happens given that you know A happened.
  - Through re-arranging, we see that $P(B|A) = \frac{P(A \cap B)}{P(B)}$.