Lecture 14 – More Combinatorics, Conditional Probability

DSC 40A, Fall 2021 @ UC San Diego

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Announcements

- Homework 6 and Survey 6 due **tonight at 11:59pm**.
  - We have a few office hours today.
- Groupwork 7 will be released today, due **Thursday 11/18 at 11:59pm**.
- Homework 7 will be released today, due **Monday 11/22 at 11:59pm**.
- Homework 5 grades are out.
- Unrelated: consider signing up for my History of Data Science seminar!

→ won't actually be 3 hours (more like 1.5 – 2)

Agenda

- A few more applications of combinatorics.
- Partitions and the law of total probability.
- Bayes' theorem.

More combinatorics

Recap

- A **sequence** is obtained by selecting k elements from a group of n possible elements with replacement, such that order matters.
  - Number of sequences: n^k.

- A **permutation** is obtained by selecting k elements from a group of n possible elements without replacement, such that order matters.
  - Number of permutations: $P(n,k) = \frac{n!}{(n-k)!}$.

- A **combination** is obtained by selecting k elements from a group of n possible elements without replacement, such that order does not matter.
  - Number of combinations: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.

Example: deck of cards

- There are 52 cards in a standard deck.
  - Each card has 1 of 4 suits (Spades, Clubs, Hearts, Diamonds).
  - Each card has 1 of 13 ~~values~~ faces (Ace, 2, 3, ..., 10, Jack, Queen, King).
  - The order of cards in a hand does not matter.
- There are 6 practice problems here; we will likely not get through them all (but solutions will be posted with the annotated slides).
- As a bonus, we will look at a code demo of how to solve all of these questions in Python, using the `itertools` library.
  - You're not required to know how this code works!

Example: deck of cards

1. How many 5 card hands are there in poker?

$\binom{52}{5}$

2. How many 5 card hands are there where all cards are of the same suit?

- choose 5 face values from 13 $\rightarrow \binom{13}{5}$
- choose 1 suit from 4 $\rightarrow \binom{4}{1}$

$\Rightarrow \binom{4}{1}\binom{13}{5}$

3. How many 5 card hands are there that include a four-of-a-kind (values aaaab, e.g. four 3s and a 5)?

13 options for a's face

48 options for b = 12 · 4

# of options for face

options for suit

$\Rightarrow \boxed{13 \cdot 12 \cdot 4}$

4. How many 5 card hands are there that have a straight, i.e. where all card values are consecutive? (e.g. 3, 4, 5, 6, 7, but the suits don't matter)

9 possible sequences:

A-5, 2-6, 3-7, 4-8, 5-9, 6-10, 7-J, 8-Q, 9-K

For each card, 4 options for suit

$$\Rightarrow \quad 9 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = \boxed{9 \cdot 4^5}$$

5. How many 5 card hands are there that are a straight flush, i.e. where all card values are consecutive and all cards are of the same suit? (e.g. 3, 4, 5, 6, 7, where all cards are diamonds)

→ still 9 possible sequences

→ 4 options for suit, but all cards are the same suit

$$\Rightarrow \quad 9 \cdot 4 = \boxed{36}$$

Total: $\binom{13}{1} \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3$

6. How many 5 card hands are there that include exactly one pair (values aabcd, e.g. two 3s, or two 5s, etc.)?

→ 13 options for the face that is repeated, choose 1 $\binom{13}{1}$

→ 4 options for the faces of the repeated cards, choose 2 $\binom{4}{2}$

→ 12 options for the faces of the remaining 3 cards, choose 3 $\binom{12}{3}$

→ 4 options for the face of each remaining card: 4^3

The law of total probability

Example: getting to school

You conduct a survey where you ask students two questions.

1. How did you get to campus today? Walk, bike, or drive? (Assume these are the only options.)
2. Were you late?

	Late	Not Late
Walk	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

	Late	Not Late
Walk	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

$P(\text{Late}) = 0.06 + 0.03 + 0.36$

$= 0.45$

Discussion Question

What's the probability that a randomly selected person was late?

A) 0.24
B) 0.30
C) 0.45
D) 0.50
E) None of the above

To answer, go to `menti.com` **and enter 5845 1569.**

Example: getting to school

	Late	Not Late
Walk	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

- Since everyone either walks, bikes, or drives to school, we have

"and"

$$P(\text{Late}) = \underbrace{P(\text{Late} \cap \text{Walk})}_{0.06} + \underbrace{P(\text{Late} \cap \text{Bike})}_{0.03} + \underbrace{P(\text{Late} \cap \text{Drive})}_{0.36}$$

	Late	Not Late
Walk	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

$$P(\text{late} \mid \text{walk}) = \frac{P(\text{late} \cap \text{walk})}{P(\text{walk})}$$

$$= \frac{0.06}{0.06 + 0.24} = \frac{0.06}{0.3} = \boxed{0.2}$$

Discussion Question

Suppose someone walked to school. What is the probability that they were late?

A) 0.06
B) 0.2
C) 0.25
D) 0.45
E) None of the above

To answer, go to `menti.com` **and enter 5845 1569.**

$$P(\text{late} \cap \text{walk}) = P(\text{walk})\, P(\text{late} \mid \text{walk})$$

Example: getting to school

	Late	Not Late
Walk	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

- Since everyone either walks, bikes, or drives to school, we have

$$P(\text{Late}) = P(\text{Late} \cap \text{Walk}) + P(\text{Late} \cap \text{Bike}) + P(\text{Late} \cap \text{Drive})$$

- Another way of expressing the same thing:

$$\begin{aligned} P(\text{Late}) = {} & P(\text{Walk})\, P(\text{Late}|\text{Walk}) + P(\text{Bike})\, P(\text{Late}|\text{Bike}) \\ & + P(\text{Drive})\, P(\text{Late}|\text{Drive}) \end{aligned}$$

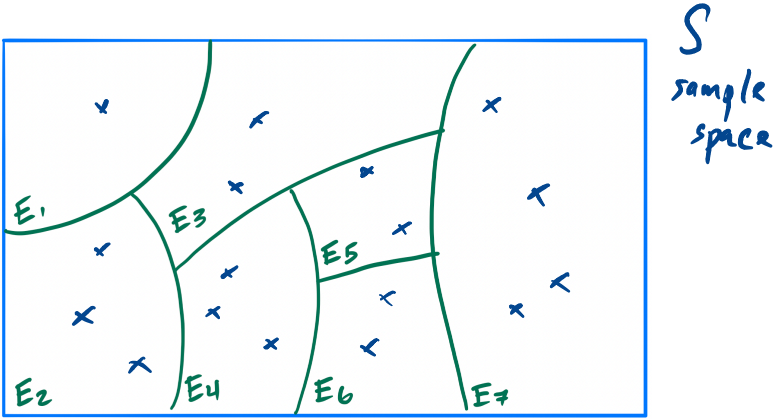
Partitions

- A set of events $E_1, E_2, ..., E_k$ is a **partition** of S if
  - $P(E_i \cap E_j) = 0$ for all unequal i, j.
  - $E_1 \cup E_2 \cup ... \cup E_k = S$. (typo in posted slides)
    - Equivalently, $P(E_1) + P(E_2) + ... + P(E_k) = 1$.
- In English, $E_1, E_2, ..., E_k$ is a partition of S if every outcome s in S is in **exactly** one event E_i.

Example partitions

- In getting to school, the events Walk, Bike, and Drive.
- In getting to school, the events Late and On-Time.
- In selecting an undergraduate student at random, the events Freshman, Sophomore, Junior, and Senior.
- In rolling a die, the events Even and Odd.
- In drawing a card from a standard deck of cards, the events Spades, Clubs, Hearts, and Diamonds.
- **Special case**: if A is an event and S is a sample space, A and $\bar{A}$ partition S.

Partitions, visualized



The law of total probability

- If A is an event and $E_1, E_2, ..., E_k$ is a **partition** of S, then

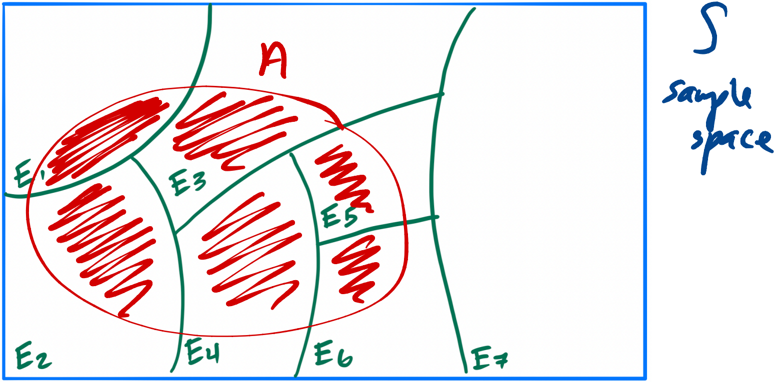
$$P(A) = P(A \cap E_1) + P(A \cap E_2) + ... + P(A \cap E_k)$$
$$= \sum_{i=1}^{k} P(A \cap E_i)$$

A : data science major

E_1 : freshman E_3 : junior

E_2 : sophomore E_4 : senior

The law of total probability, visualized



$$P(A) = P(A \cap E_1) + P(A \cap E_2) + P(A \cap E_3) + P(A \cap E_4)$$
$$+ P(A \cap E_5) + P(A \cap E_6) + P(A \cap E_7) \leftarrow 0$$

The law of total probability

- If A is an event and $E_1, E_2, \ldots, E_k$ is a **partition** of S, then

$$P(A) = P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_k)$$
$$= \sum_{i=1}^{k} P(A \cap E_i)$$

- Since $P(A \cap E_i) = P(E_i) \cdot P(A|E_i)$ by the multiplication rule, an equivalent formulation is

$$P(A) = P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2) + \ldots + P(E_k) \cdot P(A|E_k)$$
$$= \sum_{i=1}^{k} P(E_i) \cdot P(A|E_i)$$

	Late	Not Late
Walk	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

$$P(\text{walk} \mid \text{late}) = \frac{P(\text{walk} \cap \text{late})}{P(\text{late})}$$

$$= \frac{0.06}{0.45} = \frac{6}{45} \approx 0.133\ldots$$

Discussion Question

Suppose someone is late to school. What is the probability that they walked? Choose the best answer.

A) Close to 0.05
B) Close to 0.15
C) Close to 0.3
D) Close to 0.4

To answer, go to `menti.com` **and enter 5845 1569.**

Bayes' theorem

Example: getting to school

- Now suppose you don't have that entire table. Instead, all you know is
  - $P(\text{Late}) = 0.45$.
  - $P(\text{Walk}) = 0.3$.
  - $P(\text{Late}|\text{Walk}) = 0.2$.

$$P(\text{Walk} \cap \text{Late}) = P(\text{Late})\,P(\text{Walk}|\text{Late})$$
$$= P(\text{Walk})\,P(\text{Late}|\text{Walk})$$

- Can you still find $P(\text{Walk}|\text{Late})$?

$$P(\text{Walk}|\text{Late}) = \frac{P(\text{Walk} \cap \text{Late})}{P(\text{Late})} = \frac{P(\text{Walk})\,P(\text{Late}|\text{Walk})}{P(\text{Late})}$$

$$= \frac{0.3 \cdot 0.2}{0.45} = \frac{6}{45} = 0.13\overline{3}$$

Bayes' theorem (rule)

- Recall that the multiplication rule states that

$$P(A \cap B) = \underline{P(A) \cdot P(B|A)}$$

- It also states that

$$P(B \cap A) = \underline{P(B) \cdot P(A|B)}$$

- But since $A \cap B$ and $B \cap A$ are both "A and B", we have that

$$P(A) \cdot P(B|A) = P(B) \cdot P(A|B)$$

- Re-arranging yields **Bayes' theorem**:

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}$$

Bayes' theorem and the law of total probability

- Bayes' theorem:

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}$$

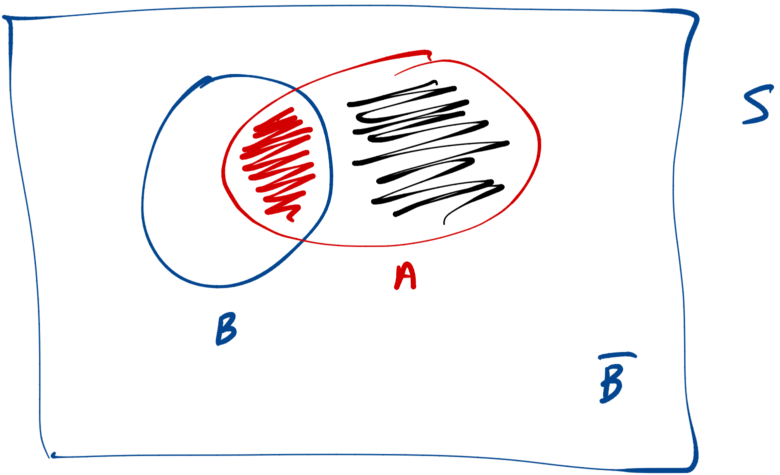
complement of B

- Recall from earlier, for any sample space S, B and $\bar{B}$ partition S. Using the law of total probability, we can re-write $P(A)$ as

$$P(A) = P(A \cap B) + P(A \cap \bar{B}) = P(B) \cdot P(A|B) + P(\bar{B}) \cdot P(A|\bar{B})$$

- This means that we can re-write Bayes' theorem as

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(B) \cdot P(A|B) + P(\bar{B}) \cdot P(A|\bar{B})}$$



$$P(A) = P(A \cap B) + P(A \cap \overline{B})$$

Example: drug testing

A manufacturer claims that its drug test will **detect steroid use 95% of the time**. What the company does not tell you is that 15% of all steroid-free individuals also test positive (the false positive rate). 10% of the Tour de France bike racers use steroids. Your favorite cyclist just tested positive. What's the probability that they used steroids?

T : test positive S : use steroids

$P(S) = 0.1$ $P(\bar{S}) = 1 - 0.1 = 0.9$

$P(T \mid S) = 0.95$

$P(T \mid \bar{S}) = 0.15$

asked for

$P(S \mid T)$

T : test positive $\quad S$: use steroids

$P(S) = 0.1 \qquad P(\bar{S}) = 1 - 0.1 = 0.9$

$P(T \mid S) = 0.95$

$P(T \mid \bar{S}) = 0.15$

asked for

$P(S \mid T)$

$$P(S \mid T) = \frac{P(S \cap T)}{P(T)} = \frac{P(S)\,P(T \mid S)}{P(S)\,P(T \mid S) + P(\bar{S})\,P(T \mid \bar{S})}$$

$$= \frac{0.1 \cdot 0.95}{0.1 \cdot 0.95 + 0.9 \cdot 0.15} = 0.41\ldots$$

Example: blind burger taste test

Want: $P(S|C)$

- Your friend claims to be able to correctly guess a burger's restaurant after just one bite.
- The probability that she correctly identifies an In-n-Out Burger is 0.55, a Shake Shack burger is 0.75, and a Five Guys burger is 0.6.
- You buy 5 In-n-Out burgers, 4 Shake Shack burgers, and 1 Five Guys burger, choose one of the burgers randomly, and give it to her.
- **Question:** Given that she guessed it correctly, what's the probability she ate a Shake Shack burger?

I : In-n-Out

S : Shake Shack

F : Five Guys

C : Guessed Correctly

$P(I) = 0.5, \quad P(S) = 0.4, \quad P(F) = 0.1$

$P(C|I) = 0.55 \quad P(C|S) = 0.75$

$P(C|F) = 0.6$

I : In-n-Out

S : Shake Shack

F : Five Guys

C : Guessed Correctly

$P(I) = 0.5, \quad P(S) = 0.4, \quad P(F) = 0.1$

$P(C|I) = 0.55 \quad P(C|S) = 0.75$

$P(C|F) = 0.6$

$$P(S|C) = \frac{P(S \cap C)}{P(C)}$$

$$= \frac{P(S)P(C|S)}{P(S)P(C|S) + P(I)P(C|I) + P(F)P(C|F)}$$

$$= \frac{0.4 \cdot 0.75}{0.4 \cdot 0.75 + 0.5 \cdot 0.55 + 0.1 \cdot 0.6} \doteq 0.47$$

$P(A|B) \stackrel{?}{=} P(A|\bar{B})$

$\Rightarrow$ Homework 7.

Discussion Question

Consider any two events A and B. Which of the following is equal to

$$P(B|A) + P(\bar{B}|A)$$

$$= \frac{P(B \cap A)}{P(A)} + \frac{P(\bar{B} \cap A)}{P(A)}$$

A) $P(A)$
B) $1 - P(B)$
C) $P(B)$
D) $P(\bar{B})$
E) 1

To answer, go to `menti.com` **and enter 5845 1569.**

$$= \frac{P(A \cap B) + P(A \cap \bar{B})}{P(A)} = \frac{P(A)}{P(A)} = 1$$

Summary

Summary

- A set of events $E_1, E_2, ..., E_k$ is a **partition** of S if each outcome in S is in exactly one E_i.
- The law of total probability states that if A is an event and $E_1, E_2, ..., E_k$ is a **partition** of S, then

$$\begin{aligned} P(A) &= P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2) + \dots + P(E_k) \cdot P(A|E_k) \\ &= \sum_{i=1}^{k} P(E_i) \cdot P(A|E_i) \end{aligned}$$

- Bayes' theorem states that

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}$$

- We often re-write the denominator $P(A)$ in Bayes' theorem using the law of total probability.