DSC 40A - Group Work Session 6

due November 18, 2022 at 11:59pm

Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. You must work in a group of 2 to 4 students for at least 50 minutes to get credit for this assignment. It's best to join a discussion section if possible.

One person from each group should submit your solutions to Gradescope by 11:59pm on **Friday (note the extended deadline)**. Make sure to **tag all group members** so everyone gets credit. This worksheet won't be graded on correctness, but rather on good-faith effort. Even if you don't solve any of the problems, you should include some explanation of what you thought about and discussed, so that you can get credit for spending time on the assignment.

Throughout this assignment, you are allowed to leave your answers in terms of factorials, the permutation formula $P(n, k)$, and the binomial coefficient $\binom{n}{k}$, unless otherwise stated.

Problem 1. Herb and Garlic

You want to plant an herb garden, so you go to a garden store that has 70 different herbs: 35 are culinary herbs, 18 are medicinal herbs, and 17 are aromatic herbs. You select 7 herbs for your herb garden by taking a random sample **without replacement** from the 70 available herbs.

a) If you consider the herbs you select as a permutation (i.e. the order in which you select each herb matters), how many permutations of 7 herbs are possible?

> **Solution:** $P(50, 5) = 254{,}251{,}200$
>
> There are $P(50, 5) = 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46$ possible sequences of 5 distinct herbs from 50 options.

b) If you consider the herbs you select as a permutation (i.e. the order in which you select each herb matters), how many permutations of 7 herbs include 4 culinary herbs and 3 aromatic herbs?

Hint: If you're stuck, you may want to complete part (d) first, then come back to this part using the relationship between permutations and combinations we discussed in lecture.

> **Solution:** $\binom{28}{2} \cdot \binom{10}{3} \cdot 5! = 28 \cdot 27 \cdot 10 \cdot 9 \cdot 8 \cdot \binom{5}{3} = 5{,}443{,}200$
>
> There are $\binom{28}{2}$ ways of picking two culinary herbs and $\binom{10}{3}$ ways of picking three aromatic herbs, and so there are
>
> $$\binom{28}{2} \cdot \binom{10}{3}$$
>
> ways of picking 2 culinary herbs and three aromatic herbs (this is the answer to part (d)). But the order matters. For each combination of 5 herbs, there are 5! different orderings. The total number of sequences is therefore
>
> $$\binom{28}{2} \cdot \binom{10}{3} \cdot 5!$$
>
> ---
>
> Here is another approach. We have to fill a sequence of 5 slots. To generate a sequence, we'll first pick the order in which the culinary herbs will appear. We'll then pick the order in which the aromatic herbs will appear. Then we will pick where the aromatic herbs will be placed in the order of five herbs.

There are $28 \cdot 27$ ways in which to order 2 culinary herbs.

There are $10 \cdot 9 \cdot 8$ ways in which to order 3 aromatic herbs.

There are $\binom{5}{3}$ ways of picking where to place the aromatic herbs in the final sequence.

The total number of sequences is

$$28 \cdot 27 \cdot 10 \cdot 9 \cdot 8 \cdot \binom{5}{3}$$

You can verify that this gives the same answer as above.

c) If you consider the herbs you select as a combination (i.e. the order in which you select each herb does not matter), how many combinations of 7 herbs are possible?

Solution: $C(50, 5) = \binom{50}{5} = 2{,}118{,}760$

There are $C(50, 5) = \binom{50}{5}$ possible sets of 5 herbs, chosen from 50.

d) If you consider the herbs you select as a combination (i.e. the order in which you select each herb does not matter), how many combinations of 7 herbs include 4 culinary herbs and 3 aromatic herbs?

Solution: $\binom{28}{2} \cdot \binom{10}{3} = 45{,}360$

There are $\binom{28}{2}$ ways to pick the culinary herbs and $\binom{10}{3}$ ways to pick the aromatic herbs, for a total of

$$\binom{28}{2} \cdot \binom{10}{3}$$

e) What is the probability that you choose 4 culinary herbs and 3 aromatic herbs for your garden?

Solution: $\dfrac{\binom{28}{2} \cdot \binom{10}{3} \cdot 5!}{P(50, 5)} = \dfrac{\binom{28}{2} \cdot \binom{10}{3}}{\binom{50}{5}} = \dfrac{162}{7567} \approx 0.021$

We can solve this probability question using a sample space of permutations or combinations. To use permutations, simply divide the answer to part (b) by the answer to part (a):

$$\frac{\binom{28}{2} \cdot \binom{10}{3} \cdot 5!}{P(50, 5)} = \frac{162}{7567} \approx 0.021$$

To use combinations, divide the answer to part (d) by the answer to part (c):

$$\frac{\binom{28}{2} \cdot \binom{10}{3}}{\binom{50}{5}} = \frac{162}{7567} \approx 0.021$$

Problem 2. Shuffling Strings

For this problem in particular, but also for the entire worksheet and the homework, you may find it helpful to refer to **these slides (link)**. However, we strongly encourage you to attempt the problem first before looking at these slides.

In this problem, a "permutation" will refer to a permutation of the entire string (i.e. we are selecting n elements from a group of n possible elements, without replacement, such that order does matter).

a) How many permutations are there of the string DOG?

> **Solution:** There are 3 options for the first character, 2 for the second, and 1 for the third — this gives 3!.

b) How many permutations are there of the string GAG?

Hint: The answer is not 6.

> **Solution:** If you enumerate all of the possibilities, you'll see there are only 3 permutations of DAD: AGG, GAG, and GGA.
>
> It's tempting to jump to the formula for permutations, which would imply that there are $3! = 6$ permutations of GAG. However, this double counts each permutation — specifically, each unique permutation is counted once for every arrangement of the two Gs.
>
> To see this concretely, suppose that instead our string is G_1AG_2. Then, there are 6 unique permutations:
>
> $$G_1AG_2 \qquad G_2AG_1 \qquad AG_1G_2 \qquad AG_2G_1 \qquad G_1G_2A \qquad G_2G_1A$$
>
> However, if G_1 and G_2 are both just treated as being Gs, then the first two permutations are the same, the second two permutations are the same, and the last two permutations are the same. So, we need to **divide** our result from using the permutation formula, 3!, by the number of arrangements of the repeated characters. In this case there are 2 repeated characters, so there are 2! ways to arrange them. Thus, the total number of permutations of DAD is $\frac{3!}{2!} = 3$.
>
> Another way to think about this problem is that we have 3 characters and need to choose 2 of them to be Gs; we can do this in $\binom{3}{2}$ ways.

c) How many permutations are there of the string GAAAGGGG?

Hint: How can you use combinations?

> **Solution:**
>
> The easiest way to think about this problem is using combinations.
>
> There are 8 positions for characters to be placed in any given permutation, and we want to choose 3 of them to be A. This can be done in $\binom{8}{3} = 56$ ways. Equivalently, we could choose 5 of the positions to be G. This can be done in $\binom{8}{5} = 56$ ways, as well.

d) How many permutations are there of the string LAJOLLA (with no spaces)?

Hint: Start by writing out the frequency of each character.

> **Solution:** LAJOLLA:
>
> - 3 Ls,
> - 2 As,
> - 1 J,
> - 1 O
>
> One way to think about this problem is to start with 7!, which would be the total number of

permutations of LAJOLLA if all characters were unique. Then, we need to account for the fact that there are repeated Ls and As, meaning that we overcounted. There are 3! ways to arrange the Ls in place and 2! ways to arrange the As in place, meaning that each unique permutation of LAJOLLA was counted $3! \cdot 2!$ times instead of once.

Therefore, the actual number of permutations of LAJOLLA is

$$\frac{7!}{3!2!}$$

Problem 3. Billy the Builder

Suppose you have 3 identical yellow blocks, 2 identical green block, 4 identical blue blocks, and 1 red block. How many different-looking towers of 10 blocks can you create by stacking all of these blocks on top of one another in one tall stack?

Hint: If you think about it a certain way, this problem is very similar to Problem 2.

Solution: $\binom{10}{2} \cdot \binom{8}{2} \cdot \binom{6}{3} \cdot \binom{3}{3}$

There are 10 blocks in total. We can create a tower of blocks by first choosing two positions out of the 10 to place the yellow blocks, then choosing two of the remaining 8 positions to place the green blocks, then choosing 3 of the remaining 6 positions to place the blue blocks, and finally choosing 3 out of the remaining 3 positions to place the red blocks. This gives

$$\binom{10}{2} \cdot \binom{8}{2} \cdot \binom{6}{3} \cdot \binom{3}{3}$$

Note that the last step can be omitted because when there are only 3 positions remaining, all of them *must* be filled with red blocks. But including this last term is also correct since $\binom{3}{3} = 1$.

There are many other ways to write the same value, because the order of the colors is arbitrary. For example, instead of choosing positions for the blocks in the order yellow, green, blue, red, we could have used the order red, yellow, blue, green, which would have resulted in the following equivalent solution:

$$\binom{10}{3} \cdot \binom{7}{2} \cdot \binom{5}{3} \cdot \binom{2}{2}$$