Lecture 8 – More Simple Linear Regression

DSC 40A, Fall 2022 @ UC San Diego

Dr. Truong Son Hy, with help from **many others**

Announcements

- Look at the readings linked on the course website!

- Groupwork Relsease Day: Thursday afternoon
  Groupwork Submission Day: Monday midnight
  Homework Release Day: Friday after lecture
  Homework Submission Day: Friday before lecture

- See `dsc40a.com/calendar` for the Office Hours schedule.

Midterm study strategy

- Review the solutions to previous homeworks and groupworks.
- Re-watch lecture, post on Campuswire, come to office hours.
- Look at the past exams at `https://dsc40a.com/resources`.
- Study in groups.
- **Remember:** it's just an exam.

Agenda

- Recap of simple linear regression.
- Correlation.
- Linear algebra review.

Recap of simple linear regression

Linear prediction rules

- **New:** Instead of predicting the same future value (e.g. salary) h for everyone, we will now use a **prediction rule** $H(x)$ that uses **features**, i.e. information about individuals, to make predictions.
- We decided to use a **linear** prediction rule, which is of the form $H(x) = w_0 + w_1 x$.
  - w_0 and w_1 are called **parameters**.

Finding the best linear prediction rule

- In order to find the best linear prediction rule, we need to pick a loss function and minimize the corresponding empirical risk.
  - We chose squared loss, $\left(y_i - H(x_i)\right)^2$, as our loss function.

- The MSE is a function R_{sq} of a function H.

$$R_{\text{sq}}(H) = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - H(x_i)\right)^2$$

- But since H is linear, we know $H(x_i) = w_0 + w_1 x_i$.

$$R_{\text{sq}}(w_0, w_1) = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - (w_0 + w_1 x_i)\right)^2$$

Finding the best linear prediction rule

- Our goal last lecture was to find the slope w_1^* and intercept w_0^* that minimize the MSE, $R_{sq}(w_0, w_1)$:

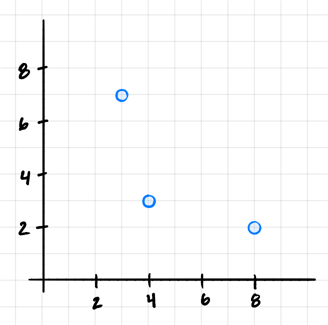
$$R_{sq}(w_0, w_1) = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - (w_0 + w_1 x_i)\right)^2$$

- We did so using multivariable calculus.

$$w_1^* = \frac{\displaystyle\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\displaystyle\sum_{i=1}^{n}(x_i - \bar{x})^2} \qquad\qquad w_0^* = \bar{y} - w_1^*\bar{x}$$

- To make predictions: $H^*(x) = w_0^* + w_1^*(x)$.

Example



$\bar{x} =$

$\bar{y} =$

$$w_1^* = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} =$$

$$w_0^* = \bar{y} - w_1^* \bar{x} =$$

x_i	y_i	$(x_i - \bar{x})$	$(y_i - \bar{y})$	$(x_i - \bar{x})(y_i - \bar{y})$	$(x_i - \bar{x})^2$
3	7				
4	3				
8	2				

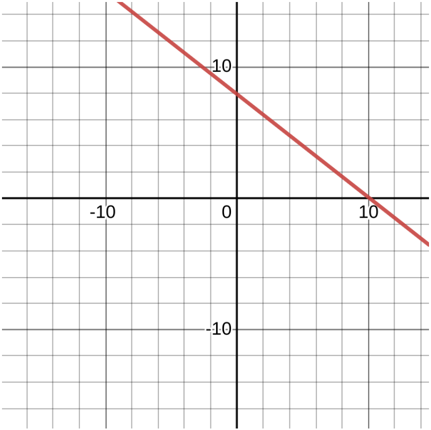
Let's solve it by computer programming!

```python
import numpy as np
def simple_lr(x, y):
    x_bar = np.mean(np.array(x))
    y_bar = np.mean(np.array(y))
    print('Table:')
    num_samples = len(x)
    sum_products = 0
    sum_squares = 0
    for i in range(num_samples):
        x_diff = x[i] - x_bar
        y_diff = y[i] - y_bar
        prod = x_diff * y_diff
        square = x_diff * x_diff
        sum_products += prod
        sum_squares += square
        print(x[i],y[i],x_diff,y_diff,prod,square)
    w1_star = sum_products / sum_squares
    w0_star = y_bar - w1_star * x_bar
    return x_bar, y_bar, w0_star, w1_star
```

```
x = [3, 4, 8]
y = [7, 3, 2]
x_bar, y_bar, w0_star, w1_star = simple_lr(x, y)

print('x_bar =', x_bar)
print('y_bar =', y_bar)
print('w1_star = ', w1_star)
print('w0_star = ', w0_star)
------------------------------------------------
Table:
3 7 -2.0 3.0 -6.0 4.0
4 3 -1.0 -1.0 1.0 1.0
8 2 3.0 -2.0 -6.0 9.0
x_bar = 5.0
y_bar = 4.0
w1_star =  -0.7857142857142857
w0_star =  7.928571428571429
```

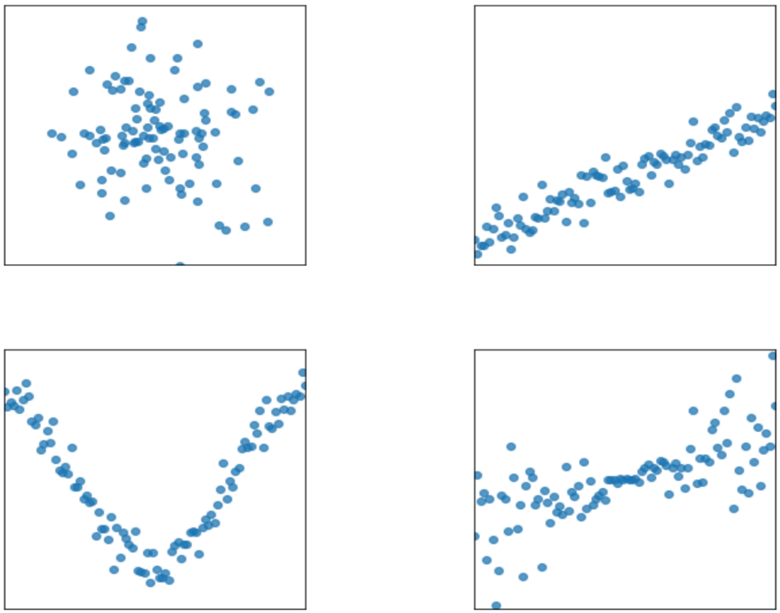
Solution to example



Terminology

- x: **features**.
- y: **response variable**.
- w_0, w_1: **parameters**.
- w_0^*, w_1^*: **optimal parameters**.
  - Optimal because they minimize mean squared error.
- The process of finding the optimal parameters for a given prediction rule and dataset is called "**fitting to the data**".
- $R_{sq}(w_0, w_1) = \frac{1}{n} \sum_{i=1}^{n} \left(y_i - (w_0 + w_1 x_i)\right)^2$: **mean squared error**, **empirical risk**.

Correlation



Correlation coefficient

- In DSC 10, you were introduced to the idea of correlation.
  - It is a measure of the strength of the **linear association** of two variables, x and y. Intuitively, it is a measure of how tightly clustered a scatter plot is around a straight line.

- The correlation coefficient, r, is defined as **the average of the product of x and y, when both are in standard units**.
  - x_i in standard units: $\dfrac{x_i - \bar{x}}{\sigma_x}$.

    y_i in standard units: $\dfrac{y_i - \bar{y}}{\sigma_y}$.

Correlation coefficient

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  - x_i in standard units: $\frac{x_i - \bar{x}}{\sigma_x}$.
    y_i in standard units: $\frac{y_i - \bar{y}}{\sigma_y}$.

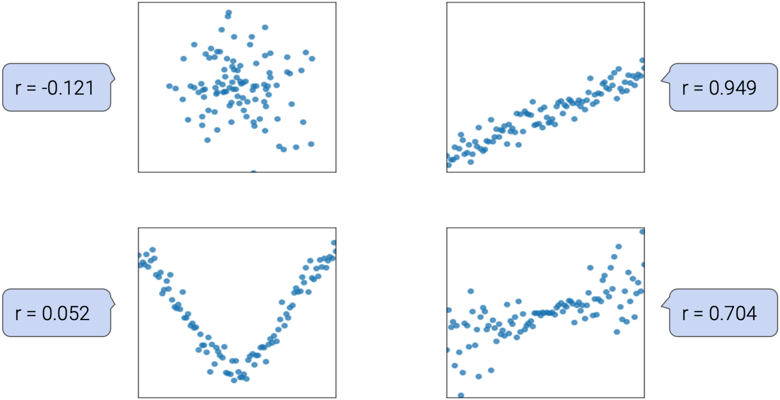
  - Definition of r:

$$r = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{x_i - \bar{x}}{\sigma_x} \right) \cdot \left(\frac{y_i - \bar{y}}{\sigma_y} \right)$$

Properties of the correlation coefficient r

- r has no units.

- It ranges between -1 and 1.
  - r = 1 indicates a perfect positive linear association (x and y lie exactly on a straight line that is sloped upwards).
  - r = –1 indicates a perfect negative linear association between x and y.
  - The closer r is to 0, the weaker the linear association between x and y is.
  - r says nothing about non-linear association.


r = -0.121
r = 0.949
r = 0.052
r = 0.704

Another way to express w_1^*

- It turns out that w_1^*, the optimal slope for the linear prediction rule, can be written in terms of r!

$$w_1^* = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = r\frac{\sigma_y}{\sigma_x}$$

- It's not surprising that r is related to w_1^*, since r is a measure of linear association.

- Concise way of writing w_0^* and w_1^*:

$$w_1^* = r\frac{\sigma_y}{\sigma_x} \qquad w_0^* = \bar{y} - w_1^*\bar{x}$$

Proof that $w_1^* = r\frac{\sigma_y}{\sigma_x}$

By definition, we have:

$$\sigma_x = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

Proof that $w_1^* = r\frac{\sigma_y}{\sigma_x}$

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Thus:

$$n\sigma_x^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2$$

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$$n\sigma_x^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2$$

On the another hand, we also have:

$$r = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{\sigma_x}\right)\cdot\left(\frac{y_i - \bar{y}}{\sigma_y}\right)$$

That leads to:

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$$rn\sigma_x\sigma_y = \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$$

Proof that $w_1^* = r\frac{\sigma_y}{\sigma_x}$ (continued)

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Therefore:

$$w_1^* = \frac{nr\sigma_x\sigma_y}{n\sigma_x^2} =$$

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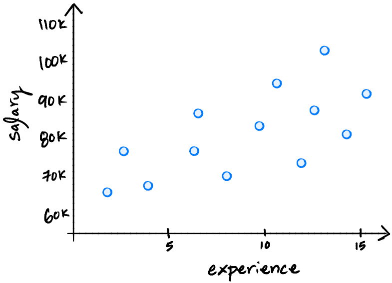
Therefore:

$$w_1^* = \frac{nr\sigma_x\sigma_y}{n\sigma_x^2} = r\frac{\sigma_y}{\sigma_x}$$

Interpreting the slope

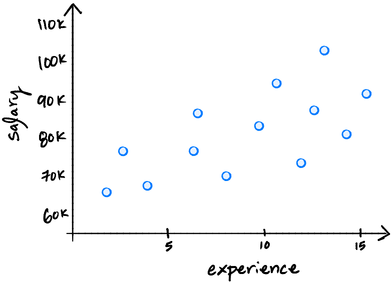
$$w_1^* = r\frac{\sigma_y}{\sigma_x}$$



- σ_y and σ_x are always non-negative. As a result, the sign of the slope is determined by the sign of r.
- As the y values get more spread out, σ_y increases and so does the slope.
- As the x values get more spread out, σ_x increases and the slope decreases.

Interpreting the intercept

$$w_0^* = \bar{y} - w_1^* \bar{x}$$



- What is $R_{sq}(H^*(\bar{x}))$?

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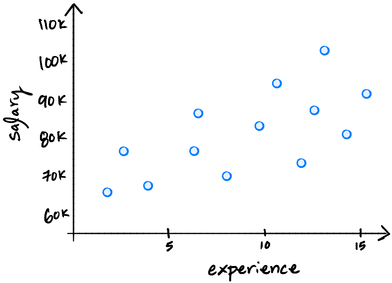

- What is $R_{sq}(H^*(\bar{x}))$?

$$R_{sq}(H^*(\bar{x})) = \frac{1}{n} \sum_{i=1}^{n} (y_i - (w_0^* + w_1^* \bar{x}))^2$$

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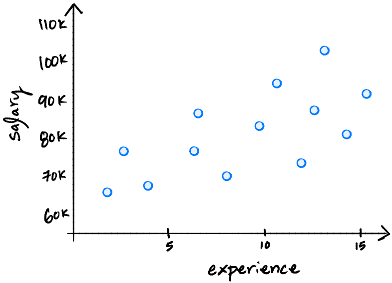

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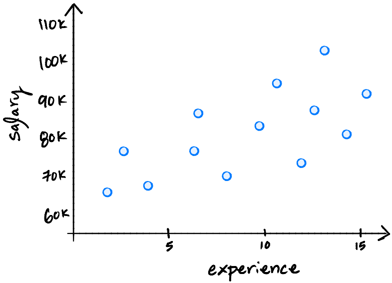

- What is $R_{sq}(H^*(\bar{x}))$?

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$$\Leftrightarrow R_{sq}(H^*(\bar{x})) = \frac{1}{n}\sum_{i=1}^{n}(y_i - (\bar{y} - w_1^* \bar{x} + w_1^* \bar{x}))^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 = \sigma_y^2$$

Actually $H^*(\bar{x}) = \bar{y}$.

Discussion Question

We fit a linear prediction rule for salary given years of experience. Then everyone gets a $5,000 raise. Which of these happens?

a) slope increases, intercept increases

b) slope decreases, intercept increases

c) slope stays same, intercept increases

d) slope stays same, intercept stays same

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Answer: C

Linear algebra review

Wait... why do we need linear algebra?

- Soon, we'll want to make predictions using more than one feature (e.g. predicting salary using years of experience and GPA).
- Thinking about linear regression in terms of **linear algebra** will allow us to find prediction rules that
  - use multiple features.
  - are non-linear.
- Before we dive in, let's review.
- **There can be linear algebra on the midterm!!**

Matrices

- An $m \times n$ **matrix** is a table of numbers with m rows and n columns.

- We use upper-case letters for matrices.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

- A^T denotes the transpose of A:

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Matrix addition and scalar multiplication

- We can add two matrices only if they are the same size.

- Addition occurs elementwise:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 8 & 9 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 8 & 10 & 12 \\ 3 & 3 & 3 \end{bmatrix}$$

- Scalar multiplication occurs elementwise, too:

$$2 \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

Matrix-matrix multiplication

- We can multiply two matrices A and B only if

  $$\#\text{ columns in } A = \#\text{ rows in } B.$$

- If A is $m \times n$ and B is $n \times p$, the result is $m \times p$.
  - This is **very useful**.

- The ij entry of the product is:

  $$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

Some matrix properties

- Multiplication is Distributive:

$$A(B + C) = AB + AC$$

- Multiplication is Associative:

$$(AB)C = A(BC)$$

- Multiplication is **not commutative**:

$$AB \neq BA$$

- Transpose of sum:

$$(A + B)^T = A^T + B^T$$

- Transpose of product:

$$(AB)^T = B^T A^T$$

Vectors

- An **vector** in $\mathbb{R}^n$ is an $n \times 1$ matrix.

- We use lower-case letters for vectors.

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 5 \\ -3 \end{bmatrix}$$

- Vector addition and scalar multiplication occur elementwise.

Geometric meaning of vectors

- A vector $\vec{v} = (v_1, \dots, v_n)$ is an arrow to the point $(v_1, \dots, v_n)$ from the origin.

- The **length**, or **norm**, of $\vec{v}$ is $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$.

Dot products

- The **dot product** of two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ is denoted by:

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$$

- Definition:

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

- The result is a **scalar**!

Discussion Question

Which of these is another expression for the length of $\vec{u}$?

a) $\vec{u} \cdot \vec{u}$

b) $\sqrt{\vec{u}^2}$

c) $\sqrt{\vec{u} \cdot \vec{u}}$

d) $\vec{u}^2$

Discussion Question

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c) $\sqrt{\vec{u} \cdot \vec{u}}$

d) $\vec{u}^2$

Answer: C

Properties of the dot product

- Commutative:

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} = \vec{u}^T \vec{v} = \vec{v}^T \vec{u}$$

- Distributive:

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

Matrix-vector multiplication

- Special case of matrix-matrix multiplication.
- Result is always a vector with same number of rows as the matrix.
- One view: a "mixture" of the columns.

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

- Another view: a dot product with the rows.

Discussion Question

If A is an $m \times n$ matrix and $\vec{v}$ is a vector in $\mathbb{R}^n$, what are the dimensions of the product $\vec{v}^T A^T A\vec{v}$?

a) $m \times n$ (matrix)
b) $n \times 1$ (vector)
c) 1×1 (scalar)
d) The product is undefined.

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Summary

Summary, next time

- The correlation coefficient, r, measures the strength of the linear association between two variables x and y.
- We can re-write the optimal parameters for the linear prediction rule (under squared loss) as

$$w_1^* = r\frac{\sigma_y}{\sigma_x} \qquad w_0^* = \bar{y} - w_1^*\bar{x}$$

- We can then make predictions using $H^*(x) = w_0^* + w_1^* x$.
- We will need linear algebra in order to generalize regression to work with multiple features.
- **Next time**: Formulate linear regression in terms of linear algebra.