Lecture 18 – Foundations of Probability

DSC 40A, Fall 2022 @ UC San Diego

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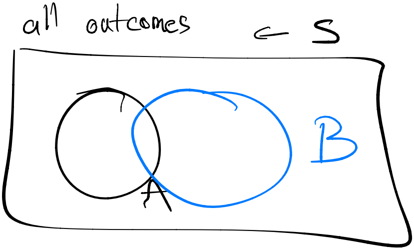
Agenda

- Conditional probability.

Principle of inclusion-exclusion

- If events A and B are not mutually exclusive, then the addition rule becomes more complicated.

A: set
B: set



$P(A \cup B) =$

- In general, if A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Discussion Question

Each day when you get home from school, there is a

- 0.3 chance your mom is at home
- 0.4 chance your brother is at home
- 0.25 chance that both your mom and brother are at home

When you get home from school today, what is the chance that neither your mom nor your brother are at home?

A) 0.3
B) 0.45
C) 0.55
D) 0.7
E) 0.75

$P(\overline{A \cup B}) = 1 - P(A \cup B) = 0.55$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.45$

To answer, go to `menti.com` **and enter 4771 9448.**

Multiplication rule and independence

- The probability that events A and B both happen is

A and B → General

$$P(A \cap B) = P(A)P(B|A)$$

$P(B|A) \neq P(B)$

- $P(B|A)$ is read "the probability that B happens, given that A happened." It is a **conditional probability**. $P(B|A)$

- If $P(B|A) = P(B)$, events A and B are **independent**.
  - Intuitively, A and B are independent if knowing that A happened gives you no additional information about event B, and vice versa.

  - For two independent events,

$$P(A \cap B) = P(A)P(B)$$

Example: rolling a die

Let's consider rolling a fair 6-sided die. The results of each die roll are independent from one another.

Mutually

- Suppose we roll the die once. What is the probability that the face is 1 and 2?

  0

- Suppose we roll the die once. What is the probability that the face is 1 or 2?

  $S = \{1, 2, 3, 4, 5, 6\}$

  $A = \{1, 2\}$

  face is either 1 or 2

  $P(A) = \frac{|A|}{|S|} = \frac{2}{6} = \frac{1}{3}$

Example: rolling a die

$P(A \cap B) = P(A)\,P(B \mid A)$

$\{1, 2, 3, 4, 5, 6\}$

$P(A \cap B \cap C) = P(A)\,P(B \mid A)\,P(C \mid A \cap B)$

- Suppose we roll the die 3 times. What is the probability that the face 1 never appears in any of the rolls?

$P(A) = 5/6$

A: First die is not 1

B: Second " " "

C: Third " " "

$P(A \cap B \cap C) = P(A)\,P(B \mid A) \times P(C \mid A \cap B) = P(A)\,P(B)\,P(C)$

$= \left(\frac{5}{6}\right)^3$

- Suppose we roll the die 3 times. What is the probability that the face 1 appears at least once?

$1 - P(A \cap B \cap C) = 1 - (5/6)^3 \rightarrow 1 - \left(1 - \frac{1}{6}\right)^3$

Mutual independence: $P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \cdots P(A_n)$

$A_1, \ldots, A_n$ are mutually independent.

pairwise independence: $P(A_i \cap A_j) = P(A_i)\,P(A_j)$

Example: rolling a die

$P(A=5) \times P(B \mid A = x)$

$1 \times \frac{1}{6} = \frac{1}{6}$

A_1: 2, 4, 6 appear on first

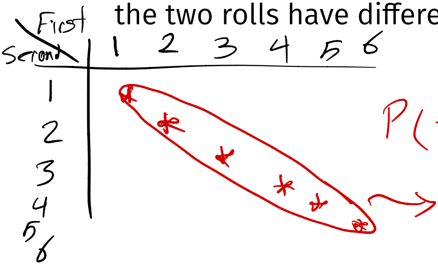
A_2: " " second . . . A_n

- Suppose we roll the die n times. What is the probability that only the faces 2, 4, and 5 appear?

$A = \{2, 4, 5\}$ $\quad P(A_1 \cap A_2 \cap \dots \cap A_n)$

$P(A) = \frac{3}{6} = \frac{1}{2}$ $\quad = P(A_1)\, P(A_2) \cdots P(A_n) = \left(\frac{1}{2}\right)^n$

- Suppose we roll the die twice. What is the probability that the two rolls have different faces?



$|S| = 36$

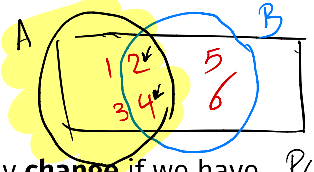
P(two rolls dif.) = 1 − P(two roll same) $= 5/6$

faces are equal $= \frac{6}{36} = \frac{1}{6}$

Conditional probability

Conditional probability

$P(B|A) = \frac{2}{4} = \frac{1}{2}$



$P(B) = \frac{2}{3}$

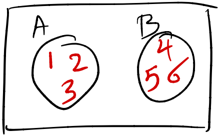
$P(B|A)$

- The probability of an event may **change** if we have additional information about outcomes.

- Starting with the multiplication rule, $P(A \cap B) = P(A)P(B|A)$, we have that

$$P(B|A) \stackrel{def}{=} \frac{P(A \cap B)}{P(A)}$$

S



assuming that $P(A) > 0$.

$P(B) = \frac{3}{6} = \frac{1}{2}$

$P(B|A) = 0$

$P(B|A) \neq P(B)$

↳ B & A are NOT independent.

Example: families

Suppose a family has two pets. Assume that it is equally likely that each pet is a dog or a cat. Consider the following two probabilities:

1. The probability that both pets are dogs given that **the oldest is a dog**.

2. The probability that both pets are dogs given that **at least one of them is a dog**.

Discussion Question

Are these two probabilities equal?

A) Yes, they're equal

B) No, they're not equal

To answer, go to `menti.com` **and enter 4771 9448.**

Example: families

Let's compute the probability that both pets are dogs given that **the oldest is a dog**.

$S = \{cd, dc, cc, dd\}$

$A =$ oldest is dog $= \{cd, dd\}$

$B =$ both dogs dogs $= \{dd\}$

$A \cap B = \{dd\}$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{1/4}{2/4} = \frac{1}{2}$$

Example: families

Let's now compute the probability that both pets are dogs given that **at least one of them is a dog**.

$$S = \{cd, dc, cc, dd\}$$

$$A = \{cd, dC, dd\} \quad \text{at least one is dog}$$

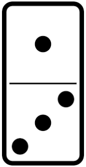
$$B = \{dd\}$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Example: dominoes (source: 538)

In a set of dominoes, each tile has two sides with a number of dots on each side: zero, one, two, three, four, five, or six. There are 28 total tiles, with each number of dots appearing alongside each other number (including itself) on a single tile.

7 cases for each side



7 x 7

0 : 7 → 7

1 : 6

⋮

6 : 1

7 + 6 + ⋯ + 1 = 28

Example: dominoes (source: 538)

Question 1: What is the probability of drawing a "double" from a set of dominoes — that is, a tile with the same number on both sides?

$S = \{ \ldots \} \Rightarrow |S| = 28$

28

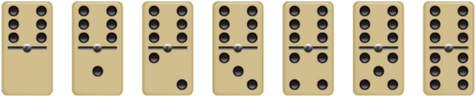
A : Domino is a double : $\{00, 11, \ldots, 66\}$

$|A| = 7$

$P(A) = \frac{7}{28} = \frac{1}{4}$

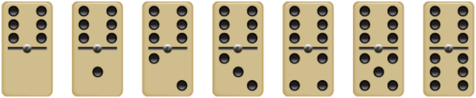
Example: dominoes (source: 538)

Question 2: Now your friend picks a random tile from the set and tells you that at least one of the sides is a 6. What is the probability that your friend's tile is a double, with 6 on both sides?



Example: dominoes (source: 538)

Question 3: Now you pick a random tile from the set and uncover only one side, revealing that it has six dots. What is the probability that this tile is a double, with six on both sides?



Simpson's Paradox (source: nih.gov)

	Treatment A	Treatment B
Small kidney stones	81 successes / 87 (93%)	234 successes / 270 (87%)
Large kidney stones	192 successes / 263 (73%)	55 successes / 80 (69%)
Combined	273 successes / 350 (78%)	289 successes / 350 (83%)

Discussion Question

Which treatment is better?

A) Treatment A for all cases.
B) Treatment B for all cases.
C) Treatment A for small stones and B for large stones.
D) Treatment A for large stones and B for small stones.

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Simpson's Paradox occurs when an association between two variables exists when the data is divided into subgroups, but reverses or disappears when the groups are combined.

- See more in DSC 80.

Summary, next time

Summary

- The probability that events A and B both happen is $P(A \cap B) = P(A)P(B|A)$.
  - $P(B|A)$ is the probability that B happens given that you know A happened.
  - Through re-arranging, we see that $P(B|A) = \frac{P(A \cap B)}{P(B)}$.