Lecture 20 – More Combinatorics, Conditional Probability

DSC 40A, Fall 2021 @ UC San Diego

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Agenda

- A few more applications of combinatorics.
- Partitions and the law of total probability.

More combinatorics

Another example

Question 2, Part 1: We have 12 pets, 5 dogs and 7 cats. In how many ways can we select 4 pets?

Another example

Question 2, Part 2: We have 12 pets, 5 dogs and 7 cats. In how many ways can we select 4 pets such that we have...

1. 2 dogs and 2 cats?
2. 3 dogs and 1 cat?
3. At least 2 dogs?

Another example

Question 2, Part 3: We have 12 pets, 5 dogs and 7 cats. We randomly select 4 pets. What's the probability that we selected at least 2 dogs?

Yet another example

Question 3: Suppose we flip a fair coin 10 times. What is the probability that we see an equal number of heads and tails?

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

One step further

Question 4: Suppose we flip a coin **that is not fair**, but instead has $P(\text{heads}) = \frac{1}{3}$, 10 times. Assume that each flip is independent.

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

Recap

- A **sequence** is obtained by selecting k elements from a group of n possible elements with replacement, such that order matters.
  - Number of sequences: n^k.

- A **permutation** is obtained by selecting k elements from a group of n possible elements without replacement, such that order matters.
  - Number of permutations: $P(n,k) = \frac{n!}{(n-k)!}$.

- A **combination** is obtained by selecting k elements from a group of n possible elements without replacement, such that order does not matter.
  - Number of combinations: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.

Example: deck of cards

- There are 52 cards in a standard deck.
  - Each card has 1 of 4 suits (Spades, Clubs, Hearts, Diamonds).
  - Each card has 1 of 13 values (Ace, 2, 3, ..., 10, Jack, Queen, King).
  - The order of cards in a hand does not matter.
- There are 6 practice problems here; we will likely not get through them all (but solutions will be posted with the annotated slides).
- As a bonus, we will look at a code demo of how to solve all of these questions in Python, using the `itertools` library.
  - You're not required to know how this code works!

Example: deck of cards

1. How many 5 card hands are there in poker?

2. How many 5 card hands are there where all cards are of the same suit?

3. How many 5 card hands are there that include a four-of-a-kind (values aaaab, e.g. four 3s and a 5)?

4. How many 5 card hands are there that have a straight, i.e. where all card values are consecutive? (e.g. 3, 4, 5, 6, 7, but the suits don't matter)

5. How many 5 card hands are there that are a straight flush, i.e. where all card values are consecutive and all cards are of the same suit? (e.g. 3, 4, 5, 6, 7, where all cards are diamonds)

6. How many 5 card hands are there that include exactly one pair (values aabcd, e.g. two 3s, or two 5s, etc.)?

The law of total probability

Example: getting to school

You conduct a survey where you ask students two questions.

1. How did you get to campus today? Walk, bike, or drive? (Assume these are the only options.)
2. Were you late?

	Late	**Not Late**
Walk	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

	Late	Not Late
Walk	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

Discussion Question

What's the probability that a randomly selected person was late?

A) 0.24
B) 0.30
C) 0.45
D) 0.50
E) None of the above

To answer, go to `menti.com` and enter 4771 9448.

Example: getting to school

	Late	Not Late
Walk	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

- Since everyone either walks, bikes, or drives to school, we have

$$P(\text{Late}) = P(\text{Late} \cap \text{Walk}) + P(\text{Late} \cap \text{Bike}) + P(\text{Late} \cap \text{Drive})$$

	Late	Not Late
Walk	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

Discussion Question

Suppose someone walked to school. What is the probability that they were late?

A) 0.06
B) 0.2
C) 0.25
D) 0.45
E) None of the above

To answer, go to `menti.com` **and enter 4771 9448.**

Example: getting to school

	Late	Not Late
Walk	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

- Since everyone either walks, bikes, or drives to school, we have

$$P(\text{Late}) = P(\text{Late} \cap \text{Walk}) + P(\text{Late} \cap \text{Bike}) + P(\text{Late} \cap \text{Drive})$$

- Another way of expressing the same thing:

$$P(\text{Late}) = P(\text{Walk})\, P(\text{Late}|\text{Walk}) + P(\text{Bike})\, P(\text{Late}|\text{Bike}) + P(\text{Drive})\, P(\text{Late}|\text{Drive})$$

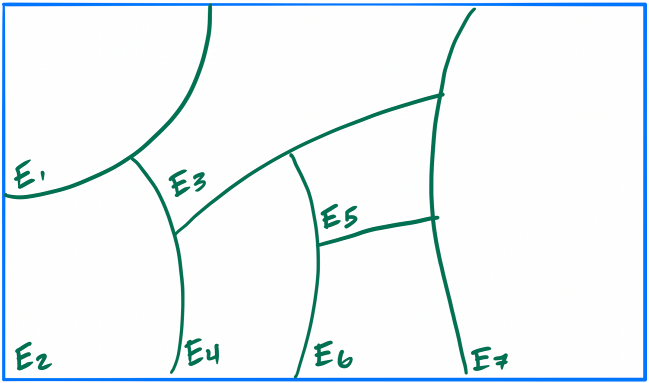
Partitions

- A set of events $E_1, E_2, \ldots, E_k$ is a **partition** of S if
  - $P(E_i \cap E_j) = 0$ for all unequal i, j.
  - $P(E_1 \cup E_2 \cup \ldots \cup E_k) = S$.
    - Equivalently, $P(E_1) + P(E_2) + \ldots + P(E_k) = 1$.
- In English, $E_1, E_2, \ldots, E_k$ is a partition of S if every outcome s in S is in **exactly** one event E_i.

Example partitions

- In getting to school, the events Walk, Bike, and Drive.
- In getting to school, the events Late and On-Time.
- In selecting an undergraduate student at random, the events Freshman, Sophomore, Junior, and Senior.
- In rolling a die, the events Even and Odd.
- In drawing a card from a standard deck of cards, the events Spades, Clubs, Hearts, and Diamonds.
- **Special case**: if A is an event and S is a sample space, A and $\bar{A}$ partition S.

Partitions, visualized

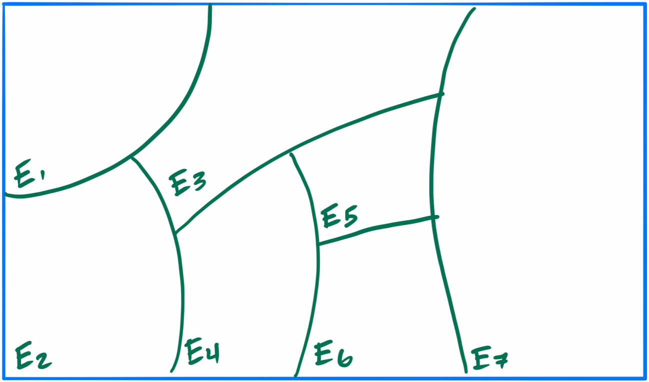


The law of total probability

- If A is an event and $E_1, E_2, ..., E_k$ is a **partition** of S, then

$$\begin{aligned} P(A) &= P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_k) \\ &= \sum_{i=1}^{k} P(A \cap E_i) \end{aligned}$$

The law of total probability, visualized



The law of total probability

- If A is an event and $E_1, E_2, ..., E_k$ is a **partition** of S, then

$$P(A) = P(A \cap E_1) + P(A \cap E_2) + ... + P(A \cap E_k)$$
$$= \sum_{i=1}^{k} P(A \cap E_i)$$

- Since $P(A \cap E_i) = P(E_i) \cdot P(A|E_i)$ by the multiplication rule, an equivalent formulation is

$$P(A) = P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2) + ... + P(E_k) \cdot P(A|E_k)$$
$$= \sum_{i=1}^{k} P(E_i) \cdot P(A|E_i)$$

	Late	Not Late
Walk	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

Discussion Question

Suppose someone is late to school. What is the probability that they walked? Choose the best answer.

A) Close to 0.05
B) Close to 0.15
C) Close to 0.3
D) Close to 0.4

To answer, go to `menti.com` **and enter 4771 9448.**

Summary

Summary

- A set of events $E_1, E_2, ..., E_k$ is a **partition** of S if each outcome in S is in exactly one E_i.
- The law of total probability states that if A is an event and $E_1, E_2, ..., E_k$ is a **partition** of S, then

$$P(A) = P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2) + ... + P(E_k) \cdot P(A|E_k)$$
$$= \sum_{i=1}^{k} P(E_i) \cdot P(A|E_i)$$