# Lecture 20 – More Combinatorics, Conditional Probability


**DSC 40A, Fall 2021 @ UC San Diego**

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# Agenda

- A few more applications of combinatorics.
- Partitions and the law of total probability.

# More combinatorics

# Another example

**Question 2, Part 1:** We have 12 pets, 5 dogs and 7 cats. In how many ways can we select 4 pets?

$\binom{12}{4}$ = # Combinations

# Another example

**Question 2, Part 2:** We have 12 pets, 5 dogs and 7 cats. In how many ways can we select 4 pets such that we have...

1. 2 dogs and 2 cats?
2. 3 dogs and 1 cat?
3. At least 2 dogs?

$$\binom{5}{2}\binom{7}{2}+\binom{5}{3}\binom{7}{1}+\binom{5}{4}$$

1 $\binom{5}{2} \times \binom{7}{2} =$ 2 dogs & 2 cats

2 $\binom{5}{3}\binom{7}{1}$

3 at least 2 dogs = 2 dogs, 2 cats OR 3 dogs, 1 cat OR 4 dogs, No cat

$\binom{5}{4} \times \binom{7}{\cancel{0}} \rightarrow 1$

# Another example

**Question 2, Part 3:** We have 12 pets, 5 dogs and 7 cats. We randomly select 4 pets. What's the probability that we selected at least 2 dogs?

A

$$P(A) = \frac{|A|}{|S|} = \frac{\binom{5}{2}\binom{7}{2} + \binom{5}{3}\binom{7}{1} + \binom{5}{4}}{\binom{12}{4}}$$

# Yet another example

H: $1/2$

T: $1/2$

**Question 3:** Suppose we flip a fair coin 10 times. What is the probability that we see an equal number of heads and tails?

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

$\binom{10}{5}\frac{1}{2^{10}}$

1. $\underbrace{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\cdots\left(\frac{1}{2}\right)} = \left(\frac{1}{2}\right)^{10}$

prob. one specific outcome

2. H T T H _ H T T T

$\binom{10}{5}$ = # cases in which I have exactly 5 H & 5 T

# One step further

**Question 4:** Suppose we flip a coin **that is not fair**, but instead has $P(\text{heads}) = \frac{1}{3}$, 10 times. Assume that each flip is independent.

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

$P(H) = \frac{1}{3}$ $\quad P(T) = \frac{2}{3}$

① : $\left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^5$

② $\underbrace{\binom{10}{5}}_{\#\text{ such outcomes}} \times \overbrace{\left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^5}^{P\{\text{specific outcome with 5 H 5 T}\}}$

# Recap

- A **sequence** is obtained by selecting $k$ elements from a group of $n$ possible elements with replacement, such that order matters.
  - Number of sequences: $n^k$.

- A **permutation** is obtained by selecting $k$ elements from a group of $n$ possible elements without replacement, such that order matters.
  - Number of permutations: $P(n, k) = \frac{n!}{(n-k)!}$.

- A **combination** is obtained by selecting $k$ elements from a group of $n$ possible elements without replacement, such that order does not matter.
  - Number of combinations: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.

# Example: deck of cards

- There are 52 cards in a standard deck.
  - Each card has 1 of 4 suits (Spades, Clubs, Hearts, Diamonds).
  - Each card has 1 of 13 values (Ace, 2, 3, ..., 10, Jack, Queen, King).
  - The order of cards in a hand does not matter.
- There are 6 practice problems here; we will likely not get through them all (but solutions will be posted with the annotated slides).
- As a bonus, we will look at a code demo of how to solve all of these questions in Python, using the `itertools` library.
  - You're not required to know how this code works!

# Example: deck of cards

1. How many 5 card hands are there in poker?

\# all cards = 52

\# combinations of 5 out of 52 $= \binom{52}{5}$

2. How many 5 card hands are there where all cards are of the same suit?

\# Combinations of 5 cards out of 13   Spades : $\binom{13}{5}$

The same is true for Clubs, Hearts & Diamonds

Overall : $4 \times \binom{13}{5}$

3. How many 5 card hands are there that include a four-of-a-kind (values aaaab, e.g. four 3s and a 5)?

There are 13 different four-of-a-kinds (AAAA, 2222, ..., KKKK)

The last card could be any OTHER card : # other cards left = $4 \times 12$

Overall : $13 \times 4 \times 12$

Note that if the smallest is $x$, the other four cards are $x+1, x+2, x+3, x+4$. We can not choose the numbers for those!

4. How many 5 card hands are there that have a straight, i.e. where all card values are consecutive? (e.g. 3, 4, 5, 6, 7, but the suits don't matter)

$\underline{4 \times 9}$ $\underbrace{\underline{4}\ \underline{4}\ \underline{4}\ \underline{4}}$ $= 9 \cdot 4^5$

smallest number

The only choice is the colors.

different colors

# cases for the card with the smallest number $= 4 \times 9$

$\{A, 2, 3, \cdots, 9\}$

* The smallest card can not be larger than 9.

5. How many 5 card hands are there that are a straight flush, i.e. where all card values are consecutive and all cards are of the same suit? (e.g. 3, 4, 5, 6, 7, where all cards are diamonds)

This, we can not choose the colors of the next 4 cards either.

$\underline{4 \times 9}$ $\underline{1}\ \underline{1}\ \underline{1}\ \underline{1}$ $= 4 \times 9$

OVERALL : $13 \times \binom{4}{2} \times \binom{12}{3} \times 4^3$

6. How many 5 card hands are there that include exactly one pair (values aabcd, e.g. two 3s, or two 5s, etc.)?

① How many ways we can pick a pair.

\# ways we pick a number × # ways we pick a pair out of 4 suits with that number

$13 \times \binom{4}{2} = 13 \times \binom{4}{2}$

② How many ways we can pick 3 other cards so that we have NO other pair?

\# ways we pick 3 other numbers out of 12 × # possible suits sequences for these 3 cards

$\binom{12}{3} \times 4^3$

# The law of total probability

# Example: getting to school

You conduct a survey where you ask students two questions.

1. How did you get to campus today? Walk, bike, or drive? (Assume these are the only options.)
2. Were you late?

| | **Late** | **Not Late** |
|---|---|---|
| **Walk** | 0.06 | 0.24 |
| **Bike** | 0.03 | 0.07 |
| **Drive** | 0.36 | 0.24 |

| | Late | Not Late |
|---|---|---|
| **Walk** | 0.06 | 0.24 |
| **Bike** | 0.03 | 0.07 |
| **Drive** | 0.36 | 0.24 |

$P(\text{late}) = 0.06 + 0.03 + 0.36 = 0.45$

## Discussion Question

What's the probability that a randomly selected person was late?

A) 0.24
B) 0.30
C) 0.45
D) 0.50
E) None of the above

**To answer, go to** `menti.com` **and enter 4771 9448.**

# Example: getting to school

| | Late | Not Late |
|---|---|---|
| **Walk** | 0.06 | 0.24 |
| **Bike** | 0.03 | 0.07 |
| **Drive** | 0.36 | 0.24 |

- Since everyone either walks, bikes, or drives to school, we have

and

$$P(\text{Late}) = P(\text{Late} \cap \text{Walk}) + P(\text{Late} \cap \text{Bike}) + P(\text{Late} \cap \text{Drive})$$

| | Late | Not Late |
|---|---|---|
| **Walk** | 0.06 | 0.24 |
| **Bike** | 0.03 | 0.07 |
| **Drive** | 0.36 | 0.24 |

## Discussion Question

Suppose someone walked to school. What is the probability that they were late?

A) 0.06
B) 0.2
C) 0.25
D) 0.45
E) None of the above

**To answer, go to** `menti.com` **and enter 4771 9448.**

# Example: getting to school

$P(A \cap B)$

$= P(B|A)$

$P(A)$

| | Late | Not Late |
|---|---|---|
| **Walk** | 0.06 | 0.24 |
| **Bike** | 0.03 | 0.07 |
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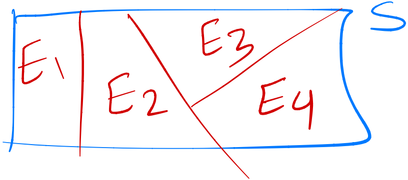
- Since everyone either walks, bikes, or drives to school, we have

$$P(\text{Late}) = P(\text{Late} \cap \text{Walk}) + P(\text{Late} \cap \text{Bike}) + P(\text{Late} \cap \text{Drive})$$

- Another way of expressing the same thing:

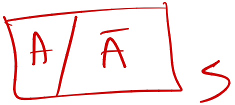
$$P(\text{Late}) = P(\text{Walk})\,P(\text{Late}|\text{Walk}) + P(\text{Bike})\,P(\text{Late}|\text{Bike}) + P(\text{Drive})\,P(\text{Late}|\text{Drive})$$

# Partitions




- A set of events $E_1, E_2, ..., E_k$ is a **partition** of $S$ if
  - $P(E_i \cap E_j) = 0$ for all unequal $i, j$.
  - $P(E_1 \cup E_2 \cup ... \cup E_k) = S$.
    - Equivalently, $P(E_1) + P(E_2) + ... + P(E_k) = 1$.
- In English, $E_1, E_2, ..., E_k$ is a partition of $S$ if every outcome $s$ in $S$ is in **exactly** one event $E_i$.

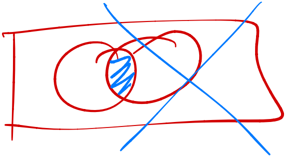
# Example partitions

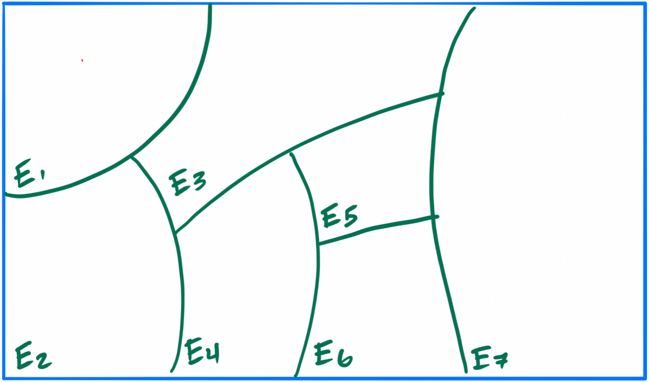



- In getting to school, the events Walk, Bike, and Drive.
- In getting to school, the events Late and On-Time.
- In selecting an undergraduate student at random, the events Freshman, Sophomore, Junior, and Senior.
- In rolling a die, the events Even and Odd.
- In drawing a card from a standard deck of cards, the events Spades, Clubs, Hearts, and Diamonds.
- **Special case**: if $A$ is an event and $S$ is a sample space, $A$ and $\bar{A}$ partition $S$.

$A, \bar{A}$

# Partitions, visualized



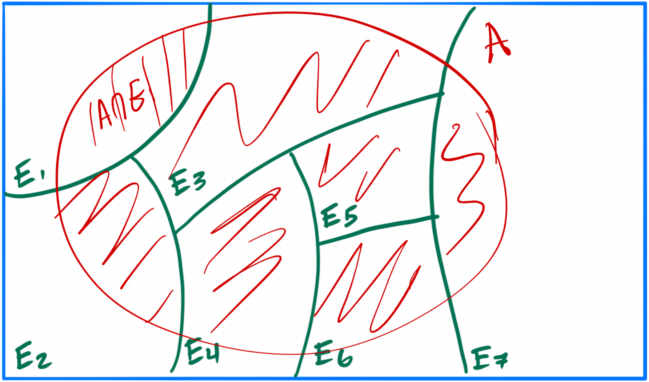


# The law of total probability

- If $A$ is an event and $E_1, E_2, \ldots, E_k$ is a **partition** of $S$, then

$$P(A) = P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_k)$$
$$= \sum_{i=1}^{k} P(A \cap E_i)$$

# The law of total probability, visualized



# The law of total probability

- If $A$ is an event and $E_1, E_2, \ldots, E_k$ is a **partition** of $S$, then

$$\begin{aligned} P(A) &= P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_k) \\ &= \sum_{i=1}^{k} P(A \cap E_i) \end{aligned}$$

- Since $P(A \cap E_i) = P(E_i) \cdot P(A|E_i)$ by the multiplication rule, an equivalent formulation is

$$\begin{aligned} P(A) &= P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2) + \ldots + P(E_k) \cdot P(A|E_k) \\ &= \sum_{i=1}^{k} P(E_i) \cdot P(A|E_i) \end{aligned}$$

| | Late | Not Late |
|---|---|---|
| **Walk** | 0.06 | 0.24 |
| **Bike** | 0.03 | 0.07 |
| **Drive** | 0.36 | 0.24 |

**Discussion Question**

Suppose someone is late to school. What is the probability that they walked? Choose the best answer.

A) Close to 0.05
B) Close to 0.15
C) Close to 0.3
D) Close to 0.4

**To answer, go to** `menti.com` **and enter 4771 9448.**

# Summary

# Summary

- A set of events $E_1, E_2, ..., E_k$ is a **partition** of $S$ if each outcome in $S$ is in exactly one $E_i$.
- The law of total probability states that if $A$ is an event and $E_1, E_2, ..., E_k$ is a **partition** of $S$, then

$$\begin{aligned} P(A) &= P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2) + ... + P(E_k) \cdot P(A|E_k) \\ &= \sum_{i=1}^{k} P(E_i) \cdot P(A|E_i) \end{aligned}$$