Lecture 20 – More Combinatorics and Introduction of Law of Total Probability

DSC 40A, Fall 2022 @ UC San Diego

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Agenda

- More Combinatorics.
- Law of Total Probability.

More Combinatorics

Summary

- A **sequence** is obtained by selecting k elements from a group of n possible elements with replacement, such that order matters.
  - Number of sequences: n^k.
- A **permutation** is obtained by selecting k elements from a group of n possible elements without replacement, such that order matters.
  - Number of permutations: $P(n,k) = \frac{n!}{(n-k)!}$.
- A **combination** is obtained by selecting k elements from a group of n possible elements without replacement, such that order does not matter.
  - Number of combinations: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.

Counting and probability

- If S is a sample space consisting of equally-likely outcomes, and A is an event, then $P(A) = \frac{|A|}{|S|}$.
- In many examples, this will boil down to using permutations and/or combinations to count $|A|$ and $|S|$.
- **Tip:** Before starting a probability problem, always think about what the sample space S is!

Discussion Question

A domino consists of two faces, each with anywhere between 0 and 6 dots. A set of dominoes consists of every possible combination of dots on each face.

How many dominoes are in the set of dominoes?

A) $\binom{7}{2}$

B) $\binom{7}{1} + \binom{7}{2}$

C) $P(7, 2)$

D) $\frac{P(7,2)}{P(7,1)} 7!$

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C) $P(7, 2)$

D) $\frac{P(7,2)}{P(7,1)} 7!$

Answer: B) $\binom{7}{1} + \binom{7}{2} = 28$.

Selecting students — overview

We're going to start by answering the same question using several different techniques.

Question 1: There are 20 students in a class. Billy is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Billy is among the 5 selected students?

Selecting students (Method 1: using permutations)

The total number of permutations is: $n! = 20!$
If Billy ranks 1st then, there are exactly $(n-1)! = 19!$ permutations so.
If Billy ranks 2nd then, there are exactly $(n-1)! = 19!$ permutations so.

...

If Billy ranks 5th then, there are exactly $(n-1)! = 19!$ permutations so.

$$|A| = 19! + 19! + 19! + 19! + 19! = 5 \cdot 19!$$

We have $|S| = 20!$. The probability is:

$$\frac{|A|}{|S|} = \frac{5 \cdot 19!}{20!} = \frac{1}{4} = 25\%$$

Selecting students (Method 2: using permutations and the complement)

If Billy ranks k-th then, there are exactly $(n-1)! = 19!$ permutations so.
Probability that Billy is not in the top 5 is:

$$\frac{\sum_{k=6}^{20} 19!}{20!} = \frac{15 \cdot 19!}{20!} = \frac{15}{20} = \frac{3}{4}$$

Probability that Billy is in the top 5 is: $1 - \frac{3}{4} = \frac{1}{4} = 25\%$.

Selecting students (Method 3: using combinations)

Question 1, Part 1 (Denominator): If you draw a sample of size 5 at random without replacement from a population of size 20, how many different **sets** of individuals could you draw?

$$|S| = \binom{20}{5}$$

Selecting students (Method 3: using combinations)

Question 1, Part 2 (Numerator): If you draw a sample of size 5 at random without replacement from a population of size 20, how many different **sets** of individuals include Billy?

Suppose Billy is already in top 5, we need to select 4 more people out of 19. Thus:

$$|A| = \binom{19}{4}$$

Selecting students (Method 3: using combinations)

The probability that Billy is in top 5 is:

$$\frac{|A|}{|S|} = \frac{\binom{19}{4}}{\binom{20}{5}} = \frac{\frac{19!}{15!4!}}{\frac{20!}{15!5!}} = \frac{5}{20} = \frac{1}{4} = 25\%$$

Selecting students (Method 4: “the easy way”)

We leave out Billy for now. There are 19 other people and 19! ways to rank them. For each way, we try to add Billy in with the top 4, and there are 5 ways to do so. By the **multiplication rule**, we have:

$$|A| = 5 \cdot 19!$$

Therefore, the final result is still: 25%.

With vs. without replacement

Discussion Question

We've determined that a probability that a random sample of 5 students from a class of 20 **without replacement** contains Billy (one student in particular) is $\frac{1}{4}$.

Suppose we instead sampled **with replacement**. Would the resulting probability be equal to, greater than, or less than $\frac{1}{4}$?

A) Equal to
B) Greater than
C) Less than

With vs. without replacement

Discussion Question

We've determined that a probability that a random sample of 5 students from a class of 20 **without replacement** contains Billy (one student in particular) is $\frac{1}{4}$.

Suppose we instead sampled **with replacement**. Would the resulting probability be equal to, greater than, or less than $\frac{1}{4}$?

A) Equal to
B) Greater than
C) Less than

Answer: C) Less than. Let's justify it mathematially!

Here, we will use the **complement rule**. We have: $|S| = 20^5$.
The number of ways to sample 5 people without ever picking up Billy is: 19^5.
The result **with replacement** becomes:

$$1 - \left(\frac{19}{20}\right)^5 \approx 22.62\% < 25\%$$

Another example

Question 2, Part 1: We have 12 pets, 5 dogs and 7 cats. In how many ways can we select 4 pets?

$$\binom{12}{4} = 495$$

Another example

Question 2, Part 2: We have 12 pets, 5 dogs and 7 cats. In how many ways can we select 4 pets such that we have...

1. 2 dogs and 2 cats?

$$\binom{5}{2} \cdot \binom{7}{2} = 10 \cdot 21 = 210$$

2. 3 dogs and 1 cat?

$$\binom{5}{3} \cdot \binom{7}{1} = 10 \cdot 7 = 70$$

3. At least 2 dogs?

$$\binom{5}{2} \cdot \binom{7}{2} + \binom{5}{3} \cdot \binom{7}{1} + \binom{5}{4} \cdot \binom{7}{0} = 210 + 70 + 5 = 285$$

Another example

Question 2, Part 3: We have 12 pets, 5 dogs and 7 cats. We randomly select 4 pets. What's the probability that we selected at least 2 dogs?

$$\frac{285}{495} \approx 57.57\%$$

Yet another example

Question 3: Suppose we flip a fair coin 10 times. What is the probability that we see an equal number of heads and tails?

1. What is the probability that we see the specific sequence THTTHTHHTH?

   $$\frac{1}{2^{10}}$$

2. What is the probability that we see an equal number of heads and tails?
   The number of ways to choose 5 out of 10 places is: $\binom{10}{5}$.
   For the chosen places, we set them 1 and the rest to be 0.
   The final result is:

   $$\frac{\binom{10}{5}}{2^{10}}$$

One step further

Question 4: Suppose we flip a coin **that is not fair**, but instead has $P(\text{heads}) = \frac{1}{3}$, 10 times. Assume that each flip is independent.

1. What is the probability that we see the specific sequence THTTHTHHTH?

$$\left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^5 = \frac{2^5}{3^{10}}$$

2. What is the probability that we see an equal number of heads and tails?

$$\binom{10}{5} \cdot \frac{2^5}{3^{10}}$$

Example: deck of cards

- There are 52 cards in a standard deck.
  - Each card has 1 of 4 suits (Spades, Clubs, Hearts, Diamonds).
  - Each card has 1 of 13 values (Ace, 2, 3, ..., 10, Jack, Queen, King).
  - The order of cards in a hand does not matter.
- There are few practice problems here; we will likely not get through them all (but solutions will be posted with the annotated slides).
- As a bonus, we will look at a code demo of how to solve all of these questions in Python, using the `itertools` library.

Example: deck of cards

1. How many 5 card hands are there in poker?

$$\binom{52}{5} = 2{,}598{,}960$$

2. How many 5 card hands are there where all cards are of the same suit?

$$4 \cdot \binom{13}{5} = 5{,}148$$

3. How many 5 card hands are there that include a four-of-a-kind (values aaaab, e.g. four 3s and a 5)?

$$13 \cdot \binom{52-4}{1} = 13 \cdot 48 = 624$$

4. How many 5 card hands are there that have a straight, i.e. where all card values are consecutive? (e.g. 3, 4, 5, 6, 7, but the suits don't matter)

$$9 \cdot 4^5 = 9,216$$

5. How many 5 card hands are there that are a straight flush, i.e. where all card values are consecutive and all cards are of the same suit? (e.g. 3, 4, 5, 6, 7, where all cards are diamonds)

$$9 \cdot 4 = 36$$

6. How many 5 card hands are there that include exactly one pair (values aabcd, e.g. two 3s, or two 5s, etc.)?

$$13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3 = 1,098,240$$

Code Demo

The law of total probability

Example: getting to school

You conduct a survey where you ask students two questions.

1. How did you get to campus today? Walk, bike, or drive? (Assume these are the only options.)
2. Were you late?

	Late	**Not Late**
Walk	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

Check sum:

$$0.06 + 0.24 + 0.03 + 0.07 + 0.36 + 0.24 = 1$$

	Late	Not Late
Walk	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

Discussion Question

What's the probability that a randomly selected person was late?

A) 0.24
B) 0.30
C) 0.45
D) 0.50
E) None of the above

	Late	Not Late
Walk	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

Discussion Question

What's the probability that a randomly selected person was late?

A) 0.24
B) 0.30
C) 0.45
D) 0.50
E) None of the above

Answer: C) 0.45

Example: getting to school

	Late	**Not Late**
Walk	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

- Since everyone either walks, bikes, or drives to school:

$$P(\text{Late}) = P(\text{Late} \cap \text{Walk}) + P(\text{Late} \cap \text{Bike}) + P(\text{Late} \cap \text{Drive})$$

$$= 0.06 + 0.03 + 0.36 = 0.45 = 45\%$$

Example: getting to school

	Late	Not Late
Walk	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

Another way of expressing the same thing:

$$
\begin{aligned}
P(\text{Late}) = {} & P(\text{Walk})\, P(\text{Late}|\text{Walk}) + P(\text{Bike})\, P(\text{Late}|\text{Bike}) \\
& + P(\text{Drive})\, P(\text{Late}|\text{Drive})
\end{aligned}
$$

$$
\begin{aligned}
P(\text{Late}) &= 0.3 \cdot \frac{0.06}{0.3} + 0.1 \cdot \frac{0.03}{0.1} + 0.6 \cdot \frac{0.36}{0.6} \\
&= 0.06 + 0.03 + 0.36 = 0.45 = 45\%
\end{aligned}
$$

	Late	Not Late
Walk	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

Discussion Question

Suppose someone walked to school. What is the probability that they were late?

A) 0.06
B) 0.2
C) 0.25
D) 0.45
E) None of the above

	Late	Not Late
Walk	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

Discussion Question

Suppose someone walked to school. What is the probability that they were late?

A) 0.06
B) 0.2
C) 0.25
D) 0.45
E) None of the above

Answer: B) 0.2

$$P(\text{Late}|\text{Walk}) = \frac{P(\text{Late} \cap \text{Walk})}{P(\text{Walk})} = \frac{P(\text{Late}, \text{Walk})}{P(\text{Walk})}$$

That is equal to:

$$P(\text{Late}|\text{Walk}) = \frac{0.06}{0.06 + 0.24} = \frac{0.06}{0.3} = \frac{1}{5} = 20\%$$