Lecture 21 – Bayes' Theorem, Independence

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Agenda

- Bayes' theorem.
- Independence.

The law of total probability

- If A is an event and $E_1, E_2, \ldots, E_k$ is a **partition** of S, then

$$\begin{aligned} P(A) &= P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_k) \\ &= \sum_{i=1}^{k} P(A \cap E_i) \end{aligned}$$

- Since $P(A \cap E_i) = P(E_i) \cdot P(A|E_i)$ by the multiplication rule, an equivalent formulation is

$$\begin{aligned} P(A) &= P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2) + \ldots + P(E_k) \cdot P(A|E_k) \\ &= \sum_{i=1}^{k} P(E_i) \cdot P(A|E_i) \end{aligned}$$

	Late	Not Late
Walk	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

Discussion Question

Suppose someone is late to school. What is the probability that they walked? Choose the best answer.

A) Close to 0.05
B) Close to 0.15
C) Close to 0.3
D) Close to 0.4

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Bayes’ theorem

Example: getting to school

- Now suppose you don't have that entire table. Instead, all you know is
  - $P(\text{Late}) = 0.45$.
  - $P(\text{Walk}) = 0.3$.
  - $P(\text{Late}|\text{Walk}) = 0.2$.
- Can you still find $P(\text{Walk}|\text{Late})$?

Bayes' theorem

- Recall that the multiplication rule states that

$$P(A \cap B) = P(A) \cdot P(B|A)$$

- It also states that

$$P(B \cap A) = P(B) \cdot P(A|B)$$

- But since $A \cap B$ and $B \cap A$ are both "A and B", we have that

$$P(A) \cdot P(B|A) = P(B) \cdot P(A|B)$$

- Re-arranging yields **Bayes' theorem**:

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}$$

Bayes' theorem and the law of total probability

- Bayes' theorem:

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}$$

- Recall from earlier, for any sample space S, B and $\bar{B}$ partition S. Using the law of total probability, we can re-write $P(A)$ as

$$P(A) = P(A \cap B) + P(A \cap \bar{B}) = P(B) \cdot P(A|B) + P(\bar{B}) \cdot P(A|\bar{B})$$

- This means that we can re-write Bayes' theorem as

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(B) \cdot P(A|B) + P(\bar{B}) \cdot P(A|\bar{B})}$$

Example: drug testing

A manufacturer claims that its drug test will **detect steroid use 95% of the time**. What the company does not tell you is that 15% of all steroid-free individuals also test positive (the false positive rate). 10% of the Tour de France bike racers use steroids. Your favorite cyclist just tested positive. What's the probability that they used steroids?

Example: blind burger taste test

- Your friend claims to be able to correctly guess a burger's restaurant after just one bite.
- The probability that she correctly identifies an In-n-Out Burger is 0.55, a Shake Shack burger is 0.75, and a Five Guys burger is 0.6.
- You buy 5 In-n-Out burgers, 4 Shake Shack burgers, and 1 Five Guys burger, choose one of the burgers randomly, and give it to her.
- **Question:** Given that she guessed it correctly, what's the probability she ate a Shake Shack burger?

Discussion Question

Consider any two events A and B. Which of the following is equal to

$$P(B|A) + P(\bar{B}|A)$$

A) $P(A)$
B) $1 - P(B)$
C) $P(B)$
D) $P(\bar{B})$
E) 1

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Example: prosecutor's fallacy

A bank was robbed yesterday by one person. Consider the following facts about the crime:

- The person who robbed the bank wore Nikes.
- Of the 10,000 other people who came to the bank yesterday, only 10 of them wore Nikes.

The prosecutor finds the prime suspect, and states that "given this evidence, the chance that the prime suspect was not at the crime scene is 1 in 1,000".

1. What is wrong with this statement?

2. Find the probability that the prime suspect is guilty given only the evidence in the exercise.

Independence

Updating probabilities

- Bayes' theorem describes how to update the probability of one event, given that another event has occurred.

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}$$

  - $P(B)$ can be thought of as the "prior" probability of B occurring, before knowing anything about A.

  - $P(B|A)$ is sometimes called the "posterior" probability of B occurring, given that A occurred.

- What if knowing that A occurred doesn't change the probability that B occurs? In other words, what if

$$P(B|A) = P(B)$$

Independent events

- A and B are **independent events** if one event occurring does not affect the chance of the other event occurring.

$$P(B|A) = P(B) \qquad\qquad P(A|B) = P(A)$$

- Otherwise, A and B are **dependent events**.

- Using Bayes' theorem, we can show that if one of the above statements is true, then so is the other.

Independent events

- **Equivalent definition:** A and B are independent events if

$$P(A \cap B) = P(A) \cdot P(B)$$

- To check if A and B are independent, use whichever is easiest:
  - $P(B|A) = P(B)$.
  - $P(A|B) = P(A)$.
  - $P(A \cap B) = P(A) \cdot P(B)$.

Mutual exclusivity and independence

Discussion Question

Suppose A and B are two events with non-zero probability.
Is it possible for A and B to be both mutually exclusive and independent?

A) Yes
B) No
C) It depends on A and B

To answer, go to `menti.com` **and enter 5686 2173.**

Example: Venn diagrams

For three events A, B, and C, we know that

- A and C are independent,
- B and C are independent,
- A and B are mutually exclusive,
- $P(A \cup C) = \frac{2}{3}$, $P(B \cup C) = \frac{3}{4}$, $P(A \cup B \cup C) = \frac{11}{12}$.

Find $P(A)$, $P(B)$, and $P(C)$.

Example: cards

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Suppose you draw two cards, one at a time.
  - A is the event that the first card is a heart.
  - B is the event that the second card is a club.

- If you draw the cards **with** replacement, are A and B independent?

- If you draw the cards **without** replacement, are A and B independent?

Example: cards

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Suppose you draw one card from a deck of 52.
  - A is the event that the card is a heart.
  - B is the event that the card is a face card (J, Q, K).
- Are A and B independent?

Assuming independence

- Sometimes we assume that events are independent to make calculations easier.
- Real-world events are almost never exactly independent, but they may be close.

Example: breakfast

1% of UCSD students are DSC majors. 25% of UCSD students eat avocado toast for breakfast. Assuming that being a DSC major and eating avocado toast for breakfast are independent:

1. What percentage of DSC majors eat avocado toast for breakfast?

2. What percentage of UCSD students are DSC majors who eat avocado toast for breakfast?

Summary

Summary

- Bayes' theorem states that

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}$$

- We often re-write the denominator $P(A)$ in Bayes' theorem using the law of total probability.
- Two events A and B are independent when knowledge of one event does not change the probability of the other event.
  - Equivalent conditions: $P(B|A) = P(B)$, $P(A|B) = P(A)$, $P(A \cap B) = P(A) \cdot P(B)$.