Lecture 22 – Independence, Conditional Independence

DSC 40A, Fall 2022 @ UC San Diego

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Agenda

- Independence.
- Conditional independence.

Independence

Updating probabilities

- Bayes' theorem describes how to update the probability of one event, given that another event has occurred.

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}$$

  - $P(B)$ can be thought of as the "prior" probability of B occurring, before knowing anything about A.
  - $P(B|A)$ is sometimes called the "posterior" probability of B occurring, given that A occurred.
- What if knowing that A occurred doesn't change the probability that B occurs? In other words, what if

$$P(B|A) = P(B)$$

A, B independent

Independent events

- A and B are **independent events** if one event occurring does not affect the chance of the other event occurring.

$$\underline{P(B|A) = P(B)} \Longrightarrow \underline{P(A|B) = P(A)}$$

- Otherwise, A and B are **dependent events**.

- Using Bayes' theorem, we can show that if one of the above statements is true, then so is the other.

$$\cancel{1\ P(B|A)} = \frac{\cancel{P(B)}\, P(A|B)}{P(A)} \Rightarrow P(A|B) = P(A)$$

Independent events

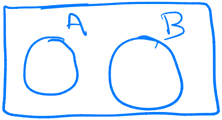
- **Equivalent definition:** A and B are independent events if

$$P(A \cap B) = P(A) \cdot P(B)$$

- To check if A and B are independent, use whichever is easiest:
  - $P(B|A) = P(B)$.
  - $P(A|B) = P(A)$.
  - $P(A \cap B) = P(A) \cdot P(B)$.

$$P(A \cap B) = P(A)\, P(B \mid A) = P(A)\, P(B)$$

Mutual exclusivity and independence



Discussion Question

Suppose A and B are two events with non-zero probability.

$A \cap B = 0$

Is it possible for A and B to be both mutually exclusive and independent?

A) Yes

B) No

C) It depends on A and B

$0 = P(A \cap B) \neq P(A)P(B) \neq 0$

$\emptyset \qquad \neq 0 \qquad \neq 0$

To answer, go to `menti.com` and enter 5686 2173.

Example: Venn diagrams

For three events A, B, and C, we know that

- A and C are independent,
- B and C are independent,
- A and B are mutually exclusive,
- $P(A \cup C) = \frac{2}{3}$, $P(B \cup C) = \frac{3}{4}$, $P(A \cup B \cup C) = \frac{11}{12}$.

Find $P(A)$, $P(B)$, and $P(C)$.

$(1)+(2)$: $a+b+2c-ac-bc=\frac{2}{3}+\frac{3}{4}$

$(1)+(2)-(3)$: $c=\frac{2}{3}+\frac{3}{4}-\frac{11}{12}=\frac{8}{12}+\frac{9}{12}-\frac{11}{12}=\frac{1}{2}$

c into (1) : $a+\frac{1}{2}-\frac{1}{2}a=\frac{2}{3} \Longrightarrow a=\frac{1}{3}$

c into (2) : $b+\frac{1}{2}-\frac{1}{2}b=\frac{3}{4} \Longrightarrow b=\frac{1}{2}$

Example: cards

A

$P(A \cap B) = P(A)P(B)$

$\boxed{♥: 2, 3, 4, 5, \cancel{6}, 7, 8, 9, 10, J, Q, K, A}$
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
$\boxed{♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A}$ B
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Suppose you draw two cards, one at a time.
  - *A* is the event that the first card is a heart.
  - *B* is the event that the second card is a club.

$P(A \cap B) = \frac{13 \times 13}{52 \times 52} = \left(\frac{1}{4}\right)^2$

- If you draw the cards **with** replacement, are *A* and *B* independent?

$P(A) = \frac{13}{52} = \frac{1}{4}$ $\quad P(B) = \frac{1}{4}$

- If you draw the cards **without** replacement, are *A* and *B* independent?

$P(B|A) = \frac{13}{51} \neq P(B) = \frac{13}{52}$

Not independent

Example: cards

B A

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Suppose you draw one card from a deck of 52.
  - A is the event that the card is a heart.
  - B is the event that the card is a face card (J, Q, K).
- Are A and B independent?

$$P(A) = \frac{13}{52} = \frac{1}{4}$$

$$P(B) = \frac{12}{52} = \frac{3}{13}$$

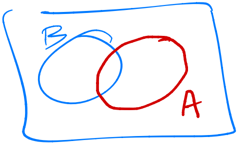
$$P(A \cap B) = \frac{3}{52}$$

$$P(A) \cdot P(B) = \frac{1}{4} \cdot \frac{3}{13} = \frac{3}{52}$$

$$P(A \cap B) = P(A) \cdot P(B)$$

independence

$$P(B) = \frac{P(A \cap B)}{P(A)}$$



the proportion of S taken by B

the proportion of A taken by B

Assuming independence

- Sometimes we assume that events are independent to make calculations easier.
- Real-world events are almost never exactly independent, but they may be close.

Example: breakfast

1% of UCSD students are DSC majors. 25% of UCSD students eat avocado toast for breakfast. Assuming that being a DSC major and eating avocado toast for breakfast are independent:

1. What percentage of DSC majors eat avocado toast for breakfast?

$P(A \mid D) = P(A) = 0.25 \rightarrow 25\%$

2. What percentage of UCSD students are DSC majors who eat avocado toast for breakfast?

A: eat Avocado
D: DSC student

$P(D \cap A) = P(D)\,P(A)$
$= 0.01 \times 0.25 = 0.0025$

Conditional independence

Conditional independence

- Sometimes, events that are dependent *become* independent, upon learning some new information.
- Or, events that are independent can become dependent, given additional information.

Example: cards

A

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

C

- Your dog ate the King of Clubs. Suppose you draw one card from a deck of 51.
  - *A* is the event that the card is a heart.
  - *B* is the event that the card is a face card (J, Q, K).
- Are *A* and *B* independent?

$P(A) = \frac{13}{51}$

$P(B) = \frac{11}{51}$

$P(A \cap B) = \frac{3}{51}$

$\frac{13}{51} \cdot \frac{11}{51} \neq \frac{3}{51}$

Example: cards

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Your dog ate the King of Clubs. Suppose you draw one card from a deck of 51.
  - A is the event that the card is a heart.
  - B is the event that the card is a face card (J, Q, K).

R

- Suppose you learn that the card is red. Are A and B independent given this new information?

$$P(A \mid R) = \frac{13}{26} = \frac{1}{2}$$

$$P(A \cap B \mid R) = \frac{3}{26}$$

$$P(B \mid R) = \frac{6}{26} = \frac{3}{13}$$

$$\frac{1}{2} \cdot \frac{3}{13} = \frac{3}{26}$$

Conditional independence

Independence $P(A \cap B) = P(A) \cdot P(B)$

- Recall that A and B are independent if

$$P(A \cap B) = P(A) \cdot P(B)$$

- A and B are **conditionally independent** given C if

$$P((A \cap B)|C) = P(A|C) \cdot P(B|C)$$

- Given that C occurs, this says that A and B are independent of one another.

$P(A \cap B) \neq P(A) P(B)$

$P(A \cap B | C) = P(A|C) \, P(B|C)$

Assuming conditional independence

- Sometimes we assume that events are conditionally independent to make calculations easier.
- Real-world events are almost never exactly conditionally independent, but they may be close.

Example: Harry Potter and TikTok

H: likes Harry ...

T: uses Tiktok

U: is a UCSD student

Suppose that 50% of UCSD students like Harry Potter and 80% of UCSD students use TikTok. What is the probability that a random UCSD student likes Harry Potter and uses TikTok, assuming that these events are conditionally independent given that a person is a UCSD student?

$$P(H \cap T \mid U) = P(H \mid U) \cdot P(T \mid U)$$

$$= 0.5 \times 0.8 = 0.4 \rightarrow 40\%$$

Independence vs. conditional independence

- Is it reasonable to assume conditional independence of
  - liking Harry Potter
  - using TikTok

  given that a person is a UCSD student?
- Is it reasonable to assume independence of these events in general, among all people?

Discussion Question

Which assumptions do you think are reasonable?

A) ~~Both~~
B) Conditional independence only
C) ~~Independence (in general) only~~
D) Neither

To answer, go to `menti.com` **and enter 5686 2173.**

Independence vs. conditional independence

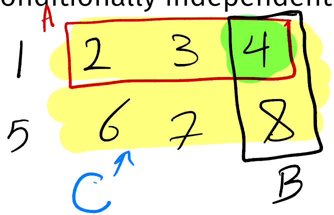
In general, there is **no relationship** between independence and conditional independence. All of these are possibilities, given three events A, B, and C.

- A and B are independent, and are conditionally independent given C.
- A and B are independent, and are conditionally dependent given C.
- A and B are dependent, and are conditionally independent given C.
- A and B are dependent, and are conditionally dependent given C.

Example: constructing events

- Consider a sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ where all outcomes are equally likely.
- For each scenario, specify events A, B, and C that satisfy the given conditions. (e.g. $A = \{2, 5, 6\}$)
- Choose events that are neither impossible nor certain, i.e. $0 < P(A), P(B), P(C) < 1$.

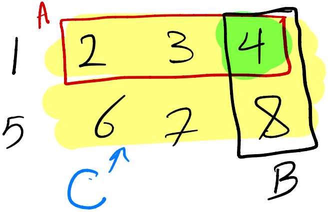
Scenario 1: A and B **are not** independent. A and B **are** conditionally independent given C.



$P(A) = \frac{3}{8}$

$P(B) = \frac{2}{8} = \frac{1}{4}$

$P(A \cap B) = \frac{1}{8}$



$$P(A|C) = \frac{3}{6} = \frac{1}{2}$$

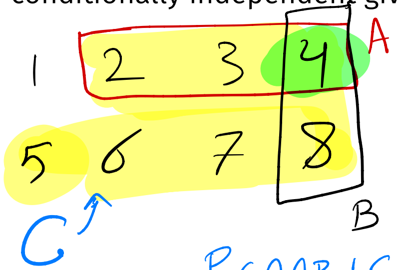
$$P(B|C) = \frac{2}{6} = \frac{1}{3}$$

$$P(A \cap B \mid C) = \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3}$$

Example: constructing events

- Consider a sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ where all outcomes are equally likely.
- For each scenario, specify events A, B, and C that satisfy the given conditions. (e.g. $A = \{2, 5, 6\}$)
- Choose events that are neither impossible nor certain, i.e. $0 < P(A), P(B), P(C) < 1$.

Scenario 2: A and B **are not** independent. A and B **are not** conditionally independent given C.



$P(A|C) = \frac{3}{7}$

$P(B|C) = \frac{2}{7}$

$P(A \cap B | C) = \frac{1}{7} \xrightarrow{} \frac{7}{49} \neq \frac{2}{7} \cdot \frac{3}{7} = \frac{6}{49}$

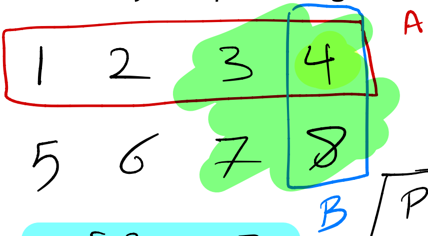
Example: constructing events

$A = \{1, 2, 3, 4\}$

$B = \{4, 8\}$

- Consider a sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ where all outcomes are equally likely.
- For each scenario, specify events A, B, and C that satisfy the given conditions. (e.g. $A = \{2, 5, 6\}$)
- Choose events that are neither impossible nor certain, i.e. $0 < P(A), P(B), P(C) < 1$.

Scenario 3: A and B **are** independent. A and B **are** conditionally independent given C.



$P(A) = \frac{4}{8} = \frac{1}{2}$

$P(B) = \frac{2}{8} = \frac{1}{4}$ ✓

$P(A \cap B) = \frac{1}{8} = \frac{1}{2} \cdot \frac{1}{4}$

$C = \{3, 4, 7, 8\}$

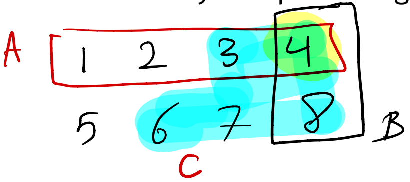
$P(A) = \frac{2}{4}$ $\quad P(A \cap B) = \frac{1}{4}$

$P(B) = \frac{2}{4}$ $\quad \frac{1}{4} = \frac{2}{4} \cdot \frac{2}{4}$

Example: constructing events

- Consider a sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ where all outcomes are equally likely.
- For each scenario, specify events A, B, and C that satisfy the given conditions. (e.g. $A = \{2, 5, 6\}$)
- Choose events that are neither impossible nor certain, i.e. $0 < P(A), P(B), P(C) < 1$.

Scenario 4: A and B **are** independent. A and B **are not** conditionally independent given C.



→ A & B are independent!

$P(A|C) = \frac{2}{5}$

$P(B|C) = \frac{2}{5}$

$P(A \cap B \mid C) = \frac{1}{5} \neq \frac{2}{5} \cdot \frac{2}{5} = \frac{4}{25}$

Recap: Bayes' theorem, independence, and conditional independence

- Bayes' theorem: $P(A|B) = \frac{P(A)P(B|A)}{P(B)}$.
- A and B are **independent** if $P(A \cap B) = P(A) \cdot P(B)$.
- A and B are **conditionally independent** given C if $P((A \cap B)|C) = P(A|C) \cdot P(B|C)$.
  - In general, there is no relationship between independence and conditional independence.
  - **See the Campuswire post on conditional independence if you're still shaky on the concept.**

Summary

Summary

- Two events A and B are conditionally independent if they are independent given knowledge of a third event, C.
  - Condition: $P((A \cap B)|C) = P(A|C) \cdot P(B|C)$.
- In general, there is no relationship between independence and conditional independence.