DSC 40A

Theoretical Foundations of Data Science I

In This Video

- More examples of using combinatorics to solve probability questions.

Counting as a Tool for Probability

Example 7. What is the probability that a randomly generated bitstring of length 10 contains an equal number of zeros and ones?

five 0's, five 1's, 10 positions

$C(10,5)$

five positions for 1's : $\{3,7,4,8,2\} \longleftrightarrow 0111001100$

$$\boxed{\frac{C(10,5)}{2^{10}}}$$

Example 8. What is the probability that a randomly generated bitstring of length 10 is the string 0011001101?

$$\frac{1}{2^{10}}$$

1 0 1 0 0 0 1 _ _ _

$$P(n,k) = (n)(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}$$

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Counting as a Tool for Probability

Example 9. What is the probability that a fair coin flipped 10 times turns up an equal number of heads and tails?

set of positions for H's $\leftarrow C(10,5)$

$\boxed{C(10,5) * \left(\frac{1}{2}\right)^{10}} = \frac{C(10,5)}{2^{10}}$

different: 6 H, 4 T

$C(10,6) = C(10,4)$

general principle:

$C(n,k) = C(n,n-k)$

Example 10. What is the probability that a fair coin flipped 10 times turns up HHTTHHTTHT?

$\frac{1}{2^{10}} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \ldots \times \frac{1}{2} = \left(\frac{1}{2}\right)^{10}$

$$P(n,k) = (n)(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Counting as a Tool for Probability

$$\boxed{C(10,5)\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^5}$$

Example 11. What is the probability that a biased coin with $Prob(H) = \frac{1}{3}$ flipped 10 times turns up an equal number of heads and tails?

S = coin toss sequences of length 10

E = coin toss sequences of length 10 with 5 H, 5 T

$$P(E) = \sum_{s \in E} p(s) = \begin{pmatrix} \#\text{ outcomes} \\ \text{in } E \\ = C(10,5) \end{pmatrix} * \left(\frac{1}{3}\right)^5 * \left(\frac{2}{3}\right)^5$$

Example 12. What is the probability that a biased coin with $Prob(H) = \frac{1}{3}$ flipped 10 times turns up $\boxed{\text{HHTTHHTTHT?}}$

$$\cancel{\frac{1}{\text{total \# coin toss sequences}} = \frac{1}{2^{10}}}$$

$$\left(\frac{1}{3}\right) * \left(\frac{1}{3}\right) * \left(\frac{2}{3}\right) * \ldots = \boxed{\left(\frac{1}{3}\right)^5 * \left(\frac{2}{3}\right)^5}$$

$$P(n,k) = (n)(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

The Easy Way

Example 6. There are 20 students in a class. A computer program selects a random sample of students by drawing 5 students at random **without replacement**. What is the chance that a particular student is among the 5 selected students?

2 ways: S = sequences

S = sets

from Theory Meets Data by Ani Adhikari, Chapter 4

The Easy Way

Example 6. There are 20 students in a class. A computer program selects a random sample of students by drawing 5 students at random **without replacement**. What is the chance that a particular student is among the 5 selected students?

17

Another way to think of sampling without replacement:

1. randomly shuffle all 20 students
2. take the first 5

$\underline{1}$ $\underline{2}$ $\underline{3}$ $\underline{4}$ $\underline{5}$ $\underline{6}$ . . . . $\underline{20}$

S = possible positions for student 17

$5/20 = \boxed{1/4}$

from Theory Meets Data by Ani Adhikari, Chapter 4

Practice Problems

Example 13. You were one of 238 individuals who reported for jury duty. If 54 of these people will be assigned to a courtroom, what is the probability that you get assigned to a courtroom?

sample without replacement

$\boxed{\frac{54}{238}}$ ← easy way

another way: S = sets of 54, chosen from 238

$C(238, 54)$ ← denominator

Practice Problems

Example 13. You were one of 238 individuals who reported for jury duty. If 54 of these people will be assigned to a courtroom, what is the probability that you get assigned to a courtroom?

How many sets of 54 individuals include you?

A. C(238, 54) C. C(238, 53)

B. C(237, 54) D. C(237, 53)

numerator

$$\boxed{\frac{C(237,53)}{C(238,54)} = \frac{54}{238}}$$

Practice Problems

Example 14. You were one of 238 individuals who reported for jury duty. Suppose 28 of these individuals are doctors. If 54 of these people will be assigned to a courtroom, what is the probability that exactly 5 doctors get assigned to a courtroom?

S = sets of 54 individuals chosen from 238

$$P(5 \text{ doctors}) = \frac{\# \text{ sets in } S \text{ with } 5 \text{ doctors}}{\# \text{ sets in } S} = \boxed{\frac{\underbrace{C(28,5)}_{\text{choose 5 doctors}} * \underbrace{C(210,49)}_{\text{choose 49 non-doctors}}}{C(238,54)}}$$

Practice Problems

Example 15. What is the probability that your five-card poker hand is a straight?

A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♡ ◇ ◇ ♤ ♡

52 cards

4 suits ◇ ♧ ♡ ♤

S = 5-card poker hands = sets of 5 cards

$$\text{prob(straight)} = \frac{\#\text{straight hands}}{\#\text{sets of 5 cards}} = \boxed{\frac{10 * 4^5}{C(52,5)}}$$

options for which 5 consecutive numbers

Practice Problems

$\diamondsuit \heartsuit \spadesuit \spadesuit$

A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

Example 16. Suppose ~~you~~ look ~~at~~ your first card as it is dealt, and you see that it is a Queen. What is the probability that your five-card hand is a straight?

S = possible sets of 4 cards that can be dealt (among 51)

$$\text{prob(straight)} = \frac{\text{\# sets of 4 cards that make straight when paired with Q}}{\cancel{\text{\# sets of 4 cards}}}$$

which #s $\rightarrow$

$$= \boxed{\frac{3 \times 4^4}{C(51,4)}}$$

$\leftarrow$ which suits

Summary

- Counting is a useful tool for probability.
- Sometimes there’s an easy way!
- **Next time:** Bayes Theorem