DSC 40A Midterm 2 Review

I'm wondering why we can't interpret the numerator as $5 * 19^4$

Practice Problems

Part 2. Numerator. If you draw a sample of size 5 at random with replacement from a population of size 20, how many different sequences of individuals include a particular person?

\# seq in 5 that include 17

ex.) 6, 4, 17, 3, 17

20

from Theory Meets Data by Ani Adhikari, Chapter 4

Hogwarts Counting Practice

Please leave your answers as unsimplified expressions with factorials, exponents, permutations, combinations, etc.

At Hogwarts School of Witchcraft and Wizardry, incoming students are each assigned to join one of the four houses: Gryffindor, Hufflepuff, Ravenclaw, and Slytherin. Suppose there are 20 incoming students, including Harry, Ron, and Hermione.

(a) All 20 incoming students get called up one at a time to wear a sorting hat that will select which house they will join. How many possible orders of all 20 students have Hermione as the first student called?

(b) How many possible orders of all 20 students have Harry, Ron, and Hermione (in any order) as the first three students called up to wear the sorting hat?

(c) How many ways are there to assign all 20 students to houses such that Harry, Ron, and Hermione all get assigned to Gryffindor?

(d) How many ways are there to assign all 20 students to houses such that exactly 4 students get assigned to Hufflepuff?

(e) How many ways are there to assign all 20 students to houses if each of the four houses has room for only five incoming students?

(f) How many ways are there to assign all 20 students to houses if nobody is assigned to the same house as the person called up to the sorting hat just before them?

(a) All 20 incoming students get called up one at a time to wear a sorting hat that will select which house they will join. How many possible orders of all 20 students have Hermione as the first student called?

> **Solution:** 19!
>
> Since we know the first student is Hermione, there are 19! ways to order the remaining students.

(b) How many possible orders of all 20 students have Harry, Ron, and Hermione (in any order) as the first three students called up to wear the sorting hat?

> **Solution:** $3! \cdot 17!$
>
> There are 3! ways to order Harry, Ron and Hermione and 17! ways to order the remaining students.

(c) How many ways are there to assign all 20 students to houses such that Harry, Ron, and Hermione all get assigned to Gryffindor?

> **Solution:** 4^{17}
>
> Harry, Ron and Hermione are already assigned and each remaining student has 4 possible houses to be assigned to. Therefore, there are 4^{17} ways to assign students to houses given the restrictions.

(d) How many ways are there to assign all 20 students to houses such that exactly 4 students get assigned to Hufflepuff?

> **Solution:** $C(20, 4)3^{16}$
>
> First, choose 4 of the 20 students to be assigned to Hufflepuff. Then each of the remaining 16 students has 3 houses they could be assigned to. Combined, this gives $C(20, 4)3^{16}$.

(e) How many ways are there to assign all 20 students to houses if each of the four houses has room for only five incoming students?

Solution: $C(20,5)C(15,5)C(10,5)C(5,5)$

First pick 5 students to be put in Gryffindor from the original 20, then pick 5 students from the remaining 15 to be put in Hufflepuff, then 5 students from the remaining 10 to be put in Ravenclaw, and lastly, 5 students from the remaining 5 to be put in Slytherin. Notice that $C(5,5) = 1$, since there are only five students left and we have to pick all of them to be in Slytherin, so we don't really have any decisions to make.

(f) How many ways are there to assign all 20 students to houses if nobody is assigned to the same house as the person called up to the sorting hat just before them?

Solution: $4 \cdot 3^{19}$

The first person can be assigned to any of the 4 houses, but all successive people can only be assigned to 3 houses since they cannot be in the same house as the person before them. This gives $4 \cdot 3^{19}$.

678-999-8212 (10 digits)

7. Soulja Boy Tell 'Em [12 Points]

In this question, we'll consider the phone number 6789998212 (mentioned in Soulja Boy's 2008 classic, "Kiss Me thru the Phone").

Note: In this problem, you may leave your answers in terms of $\binom{n}{k}$, $P(n,k)$, and/or factorials. However, you must provide a one-sentence explanation for your answer in each subpart.

AGGRAVATE

8 7 9 8 2 9 6 9 2 1

- **a)** [3 Points] How many permutations of 6789998212 are there?
- **b)** [3 Points] How many permutations of 6789998212 have all three 9s next to each other?
- **c)** [3 Points] How many permutations of 6789998212 end with a 1 and start with a 6?

  C(3,2)·1

  C(8,3) * C(5,2) * 1

- **d)** [3 Points] How many different 3 digit numbers with unique digits can we create by selecting digits from 6789998212?

(a) $C(10,3) * C(7,2) * C(5,2) * 3 * 2 * 1$

where do 9's go?

ex) GGA

GAG

AGG

C(3,2)

$8 * C(7,2) * C(5,2) \cdot 3 \cdot 2 \cdot 1$

choose 3 adjacent for 9's to go

prob. roadmap: adjacent queens in a shuffled deck of cards

51

687-999-8212

$\overbrace{\{1, 2, 6, 7, 8, 9\}}^{6 \text{ \#s}}$

3-digit #s with unique digits

$$P(6, 3) = \underline{6} \cdot \underline{5} \cdot \underline{4}$$

7. Soulja Boy Tell 'Em [12 Points]

In this question, we'll consider the phone number 6789998212 (mentioned in Soulja Boy's 2008 classic, "Kiss Me thru the Phone").

Note: In this problem, you may leave your answers in terms of $\binom{n}{k}$, $P(n, k)$, and/or factorials. However, you must provide a one-sentence explanation for your answer in each subpart.

a) [3 Points] How many permutations of 6789998212 are there?

Solution: If all digits in 6789998212 were unique, there would be 10! permutations. However, some of the digits are repeated; for each repeated digit, we need to divide by the number of ways to re-arrange the repeated digits amongst themselves.

Let's count the number of occurrences of each digit.

- 6: 1 occurrence
- 7: 1 occurrence
- 8: 2 occurrences
- 9: 3 occurrences
- 2: 2 occurrences
- 1: 1 occurrence

Thus, the number of permutations of 6789998212 is

$$\frac{10!}{2!3!2!}$$

Another way to arrive at this result is by saying first, we need to choose 1 of the 10 positions to contain a 1. Then, we need to choose 1 of the remaining 9 positions to contain a 7. Then, we need to choose 2 of the remaining 8 positions to contain a 8, and so on and so forth. This method yields

$$\binom{10}{1}\binom{9}{1}\binom{8}{2}\binom{6}{3}\binom{3}{2}\binom{1}{1}$$

If you expand out this result, you'll see that it's equal to the first result. You may also notice that this is equivalent to $\binom{10}{3}\binom{7}{2}\binom{5}{2}\binom{3}{1}\binom{2}{1}\binom{1}{1}$, which follows the same approach but chooses locations for the repeated characters first.

b) [3 Points] How many permutations of 6789998212 have all three 9s next to each other?

Solution: The key insight here is that we can treat all 3 9s as being just a single character, X, and determine the number of permutations of the 8-character string 678X8212. To do so, we again count the number of occurrences of each digit:

- 6: 1 occurrence
- 7: 1 occurrence
- 8: 2 occurrences
- X: 1 occurrence
- 2: 2 occurrences
- 1: 1 occurrence

Thus, the number of permutations of 6789998212 with all 3 9s appearing together is

$$\frac{8!}{2!2!}$$

c) [3 Points] How many permutations of 6789998212 end with a 1 and start with a 6?

Solution: Similarly to the previous part, we can "fix" the 1 at the start and 6 at the end. All we really need to determine is the number of permutations of 78999822, which is

$$\frac{8!}{2!3!2!}$$

d) [3 Points] How many different 3 digit numbers with unique digits can we create by selecting digits from 6789998212?

Solution:

The key here is that the 3 digit numbers that we're creating must have unique digits. There are 6 unique digits in 6789998212: 6, 7, 8, 9, 2, 1.

There are 6 options for the first digit of our 3 digit number, 5 options for the second digit, and 4 options for the third digit. Thus, the number of 3 digit numbers we can create in this way is $6 \cdot 5 \cdot 4 = 120$. This is also $P(6, 3)$.

$P(A|C) = P(A)$

$P(B|A) = \frac{P(A|B) * P(B)}{P(A)}$

8. Declaration of Independence [9 Points]

Suppose you're given the following probabilities:

- $P(A|B) = \frac{2}{5}$
- $P(B|A) = \frac{1}{4}$
- $P(A|C) = \frac{2}{3}$

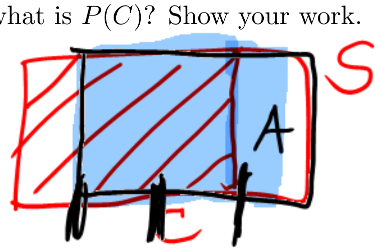
$\boxed{P(A) = \frac{2}{3}}$

$=$

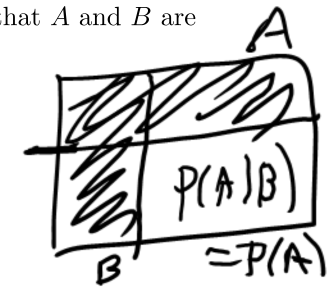
a) [3 Points] If A and C are independent, what is $P(B)$? Show your work.

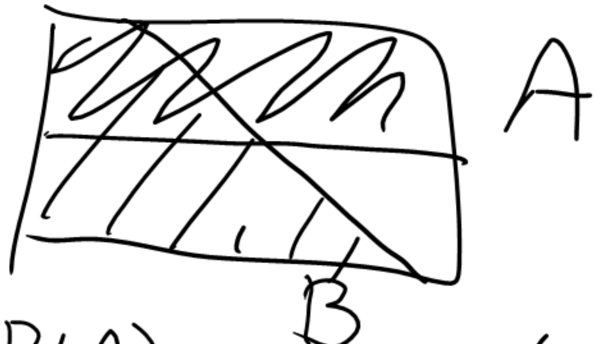
b) [6 Points] Suppose A and C are not independent, and now suppose that $P(A|\overline{C}) = \frac{1}{5}$. Given that A and B are independent, what is $P(C)$? Show your work.

A takes up 2/3 of C



$P(A|C) = 2/3)$





$P(A) = 1/2$, $P(A|B) < 1/2$

8. Declaration of Independence [9 Points]

Suppose you're given the following probabilities:

- $P(A|B) = \frac{2}{5}$
- $P(B|A) = \frac{1}{4}$
- $P(A|C) = \frac{2}{3}$

a) [3 Points] If A and C are independent, what is $P(B)$? Show your work.

> **Solution:** If A and C are independent, then $P(A) = P(A|C) = \frac{2}{3}$. Then, from Bayes' rule, we have
>
> $$P(A|B) = \frac{P(A)P(B|A)}{P(B)} \implies P(B) = \frac{P(A)P(B|A)}{P(A|B)} = \frac{\frac{2}{3} \cdot \frac{1}{4}}{\frac{2}{5}} = \frac{5}{12}$$

b) [6 Points] Suppose A and C are not independent, and now suppose that $P(A|\overline{C}) = \frac{1}{5}$. Given that A and B are independent, what is $P(C)$? Show your work.

> **Solution:**
>
> Since we're given that A and B are independent, $P(A) = P(A|B) = \frac{2}{5}$. By the Law of Total Probability, we have
>
> $$P(A) = P(A \cap C) + P(A \cap \overline{C}) = P(C)P(A|C) + P(\overline{C})P(A|\overline{C})$$
>
> For simplicity, let $p = P(C)$. Then, substituting in our known values $P(A)$, $P(A|C)$, and $P(A|\overline{C})$ and solving for p yields
>
> $$\begin{aligned} P(A) &= P(A \cap C) + P(A \cap \overline{C}) = P(C)P(A|C) + P(\overline{C})P(A|\overline{C}) \\ \frac{2}{5} &= \frac{2}{3}p + \frac{1}{5}(1-p) \\ \frac{2}{5} &= \left(\frac{10}{15} - \frac{3}{15}\right)p + \frac{1}{5} \\ \frac{1}{5} &= \frac{7}{15}p \\ p &= \frac{15}{7} \cdot \frac{1}{5} = \frac{3}{7} \end{aligned}$$
>
> Thus, $P(C) = \frac{3}{7}$.

a) 🥑🥑🥑 Suppose we draw a uniform random sample, **with replacement**, of size k from a population of n individuals. What is the probability that the sample includes some individual more than once?

there is a repeat

b) 🥑🥑🥑 Suppose we draw a uniform random sample, **without replacement**, of size k from a population of n individuals. One of the people in the random sample is Fred. A bootstrap sample (k draws at random with replacement) is drawn from the sample. Find the probability that Fred is not in the bootstrap sample.

a) complement: choose k distinct individuals (nobody appears more than once)

$$\frac{k}{n} \times \frac{k-1}{n} \times \cdots$$

$$\frac{n}{n} \times \frac{n-1}{n} \times \frac{n-2}{n} \cdots \frac{n-(k-1)}{n} = \frac{\frac{n!}{(n-k)!}}{n^k}$$

$p1 \quad p2 \quad p3 \quad \cdots \quad pk \quad = \frac{P(n,k)}{n^k}$

answer: $\boxed{1 - \frac{P(n,k)}{n^k}}$

a) Suppose we draw a uniform random sample, **with replacement**, of size k from a population of n individuals. What is the probability that the sample includes some individual more than once?

b) Suppose we draw a uniform random sample, **without replacement**, of size k ($=20$) from a population of n ($=100$) individuals. One of the people in the random sample is Fred. A bootstrap sample (k draws at random with replacement) is drawn from the sample. Find the probability that Fred is not in the bootstrap sample.

$20 = k$

b) k people, one is Fred

pick k at random with replacement

$$P(\text{no Fred}) = \left(\frac{k-1}{k}\right)^k$$

ex.) $\left(\frac{19}{20}\right)^{20}$