

Lecture 4 – Center and Spread, Other Loss Functions



DSC 40A, Spring 2023

Announcements

- ▶ Homework 1 is due **tomorrow at 11:59pm**.
 - ▶ LaTeX template provided if you want to type your answers.
 - ▶ Make sure to explain your answers! Don't just write a number; show how you got it.
 - ▶ Please come to office hours!
- ▶ Discussion section is on Wednesday.

Agenda

- ▶ Recap of empirical risk minimization.
- ▶ Center and spread.
- ▶ A new loss function.

Recap of empirical risk minimization

Empirical risk minimization

- ▶ **Goal:** Given a dataset y_1, y_2, \dots, y_n , determine the best prediction h^* .
- ▶ Strategy:
 1. Choose a **loss function**, $L(h, y)$, that measures how far any particular prediction h is from the “right answer” y .
 2. Minimize **empirical risk** (also known as average loss) over the entire dataset. The value(s) of h that minimize empirical risk are the resulting “best predictions”.

$$R(h) = \frac{1}{n} \sum_{i=1}^n \underline{L(h, y_i)}$$

Absolute loss and squared loss

- ▶ General form of empirical risk:

$$R(h) = \frac{1}{n} \sum_{i=1}^n L(h, y_i)$$

- ▶ **Absolute loss:** $L_{\text{abs}}(h, y) = |y - h|$.
 - ▶ Empirical risk: $R_{\text{abs}}(h) = \frac{1}{n} \sum_{i=1}^n |y_i - h|$. Also called “**mean absolute error**”.
 - ▶ Minimized by $h^* = \text{Median}(y_1, y_2, \dots, y_n)$.
- ▶ **Squared loss:** $L_{\text{sq}}(h, y) = (y - h)^2$.
 - ▶ Empirical risk: $R_{\text{sq}}(h) = \frac{1}{n} \sum_{i=1}^n (y_i - h)^2$. Also called “**mean squared error**”.
 - ▶ Minimized by $h^* = \text{Mean}(y_1, y_2, \dots, y_n)$.

$$y_1 = 1, y_2 = 5 \quad h=2$$

Discussion Question

Consider a dataset y_1, y_2, \dots, y_n .

Recall,

$$R_{abs}(h) = \frac{1}{n} \sum_{i=1}^n |y_i - h|$$

$$R_{abs}(2) = \frac{1}{2}(1+3) = 2$$

$$R_{sq}(h) = \frac{1}{n} \sum_{i=1}^n (y_i - h)^2$$

$$R_{sq}(2) = \frac{1}{2}(1+9) = 5$$

Is it true that, for any h , $[R_{abs}(h)]^2 = R_{sq}(h)$?

- a) True
- b) False

$$x^2 + y^2 + z^2 \neq (x + y + z)^2$$

to prove something false, give counterexample

$$2^2 \neq 5$$



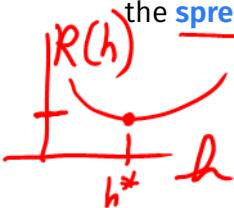
Center and spread

What does it mean?

- ▶ General form of empirical risk:

$$R(h) = \frac{1}{n} \sum_{i=1}^n L(h, y_i)$$

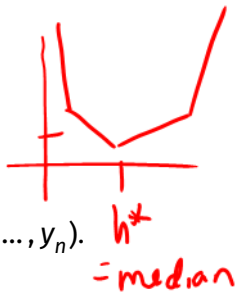
- ▶ The input h^* that minimizes $R(h)$ is some measure of the **center** of the data set.
 - ▶ e.g. median, mean, mode.
- ▶ The minimum output $R(h^*)$ represents some measure of the **spread**, or variation, in the data set.



Absolute loss

- ▶ The empirical risk for the absolute loss is

$$R_{abs}(h) = \frac{1}{n} \sum_{i=1}^n |y_i - h|$$



- ▶ $R_{abs}(h)$ is minimized at $h^* = \text{Median}(y_1, y_2, \dots, y_n)$.

- ▶ Therefore, the minimum value of $R_{abs}(h)$ is

$$\begin{aligned} \underline{R_{abs}(h^*)} &= R_{abs}(\text{Median}(y_1, y_2, \dots, y_n)) \\ &= \frac{1}{n} \sum_{i=1}^n |y_i - \text{Median}(y_1, y_2, \dots, y_n)|. \end{aligned}$$

spread { avg distance to median,
for all data points }

Mean absolute deviation from the median

- ▶ The minimum value of $R_{abs}(h)$ is the **mean absolute deviation from the median**.

$$\frac{1}{n} \sum_{i=1}^n |y_i - \text{Median}(y_1, y_2, \dots, y_n)|$$

- ▶ It measures how far each data point is from the median, on average.



Discussion Question

For the data set 2, 3, 3, 4, what is the mean absolute deviation from the median? ≈ 3

1, 0, 0, 1

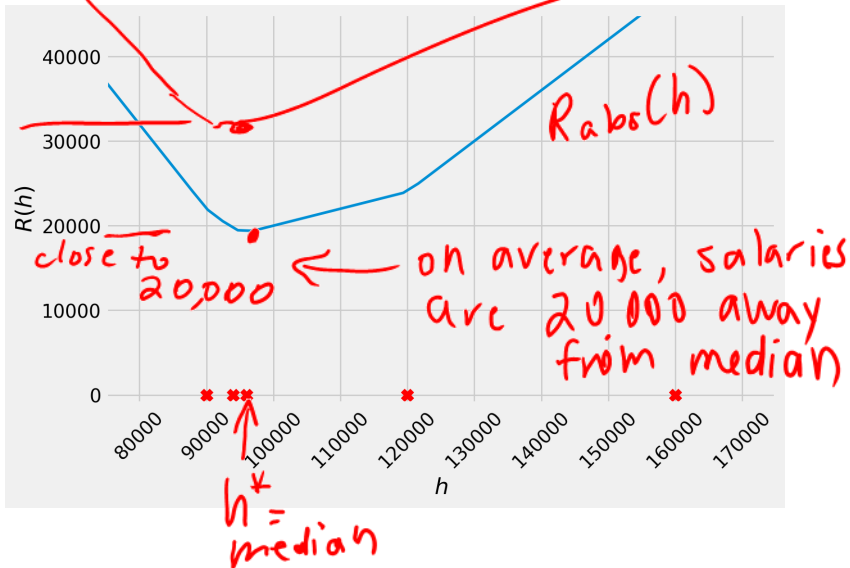
a) 0

b) $\frac{1}{2}$

c) 1

d) 2

Mean absolute deviation from the median



Squared loss

- ▶ The empirical risk for the squared loss is

$$R_{\text{sq}}(h) = \frac{1}{n} \sum_{i=1}^n (y_i - h)^2$$

↑ mean

- ▶ $R_{\text{sq}}(h)$ is minimized at $h^* = \text{Mean}(y_1, y_2, \dots, y_n)$.
- ▶ Therefore, the minimum value of $R_{\text{sq}}(h)$ is

$$\begin{aligned} R_{\text{sq}}(h^*) &= R_{\text{sq}}(\text{Mean}(y_1, y_2, \dots, y_n)) \\ &= \frac{1}{n} \sum_{i=1}^n (y_i - \text{Mean}(y_1, y_2, \dots, y_n))^2. \end{aligned}$$

measures
spread

{ variance

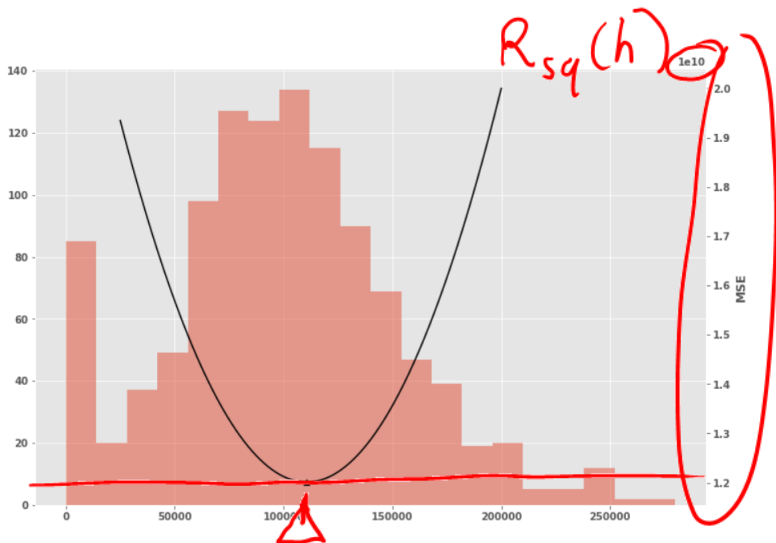
Variance

- ▶ The minimum value of $R_{sq}(h)$ is the mean squared deviation from the mean, more commonly known as the **variance**.

$$\frac{1}{n} \sum_{i=1}^n (y_i - \text{Mean}(y_1, y_2, \dots, y_n))^2$$

- ▶ It measures the squared distance of each data point from the mean, on average.
- ▶ Its square root is called the **standard deviation**.

Variance



variance is $\approx 1.2 \times 10^{10}$

mean salary was about \$110,000

0-1 loss

ex.)



- ▶ The empirical risk for the 0-1 loss is

$$R_{0,1}(h) = \frac{1}{n} \sum_{i=1}^n \begin{cases} 0, & \text{if } h = y_i \\ 1, & \text{if } h \neq y_i \end{cases}$$

mode

$R_{0,1}(h^*)$

$$= \frac{4}{7}$$

- ▶ This is the proportion (between 0 and 1) of data points not equal to h .

- ▶ $R_{0,1}(h)$ is minimized at $h^* = \text{Mode}(y_1, y_2, \dots, y_n)$.
- ▶ Therefore, $R_{0,1}(h^*)$ is the proportion of data points not equal to the mode.

A poor way to measure spread

- ▶ The minimum value of $R_{0,1}(h)$ is the proportion of data points not equal to the mode.
- ▶ A higher value means less of the data is clustered at the mode.
- ▶ Just as the mode is a very simplistic way to measure the center of the data, this is a very crude way to measure spread.

Summary of center and spread

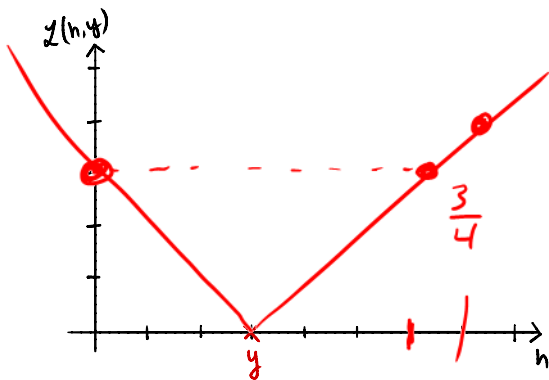
- ▶ Different loss functions lead to empirical risk functions that are minimized at various measures of center.
- ▶ The minimum values of these risk functions are various measures of spread.
- ▶ There are many different ways to measure both center and spread. These are sometimes called **descriptive statistics**.

Summary
statistics

A new loss function

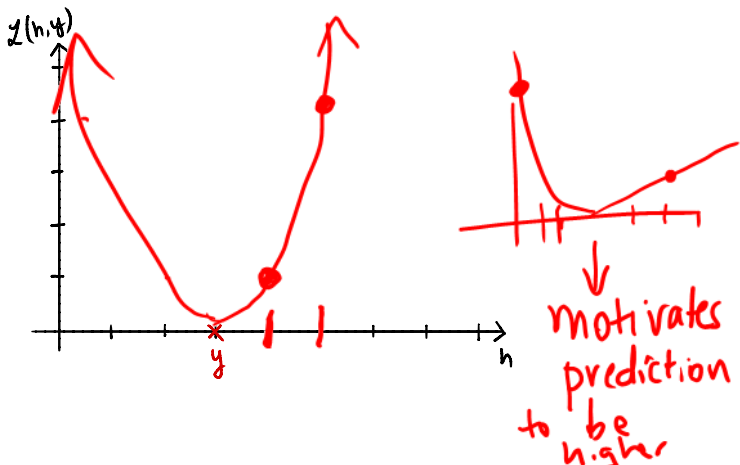
Plotting a loss function

- ▶ The plot of a loss function tells us how it treats outliers.
- ▶ Consider y to be some fixed value. Plot $L_{\text{abs}}(h, y) = \underline{|y - h|}$:



Plotting a loss function

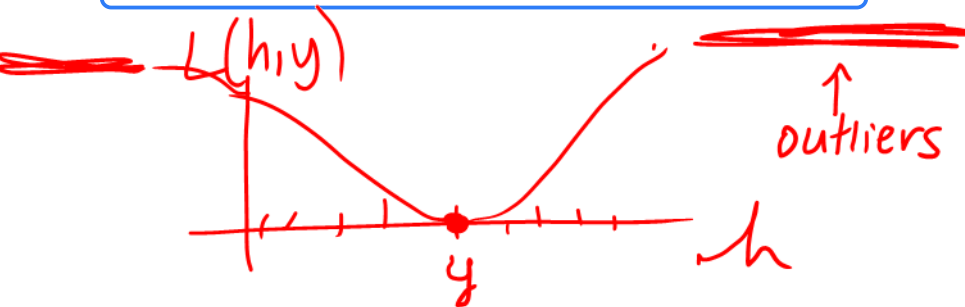
- ▶ The plot of a loss function tells us how it treats outliers.
- ▶ Consider y to be some fixed value. Plot $L_{\text{sq}}(h, y) = (y - h)^2$:



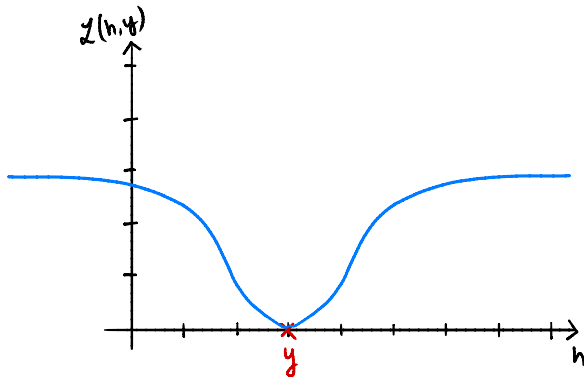
Discussion Question

Suppose L considers all outliers to be equally bad. What would it look like far away from y ?

- a) flat
- b) rapidly decreasing
- c) rapidly increasing



A very insensitive loss



- ▶ We'll call this loss L_{ucsd} because we made it up at UCSD.

Discussion Question

Which of these could be $L_{ucsd}(h, y)$?

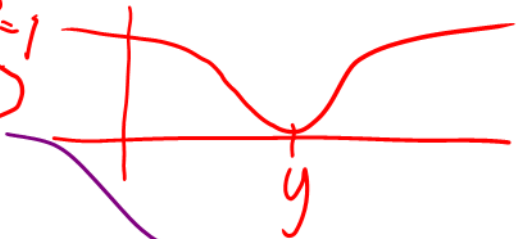
~~a) $e^{-(y-h)^2}$~~

$h=y$
 \downarrow
 $e^0 = 1$

b) $1 - e^{-(y-h)^2}$

~~c) $1 - (y-h)^2$~~

d) $1 - e^{-|y-h|}$



When $h=y$, must be 0

→ try it
on
desmos!

Adding a scale parameter

- ▶ Problem: L_{ucsd} has a fixed scale. This won't work for all datasets.
 - ▶ If we're predicting temperature, and we're off by 100 degrees, that's bad.
 - ▶ If we're predicting salaries, and we're off by 100 dollars, that's pretty good.
 - ▶ What we consider to be an outlier depends on the scale of the data.

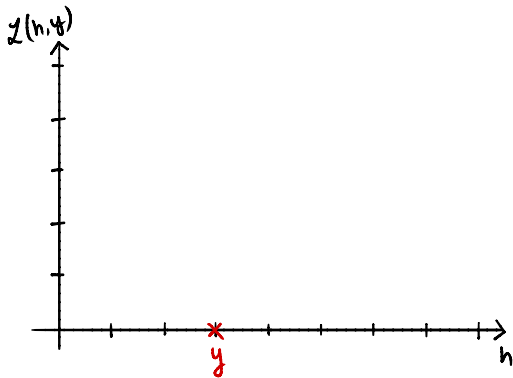
- ▶ Fix: add a **scale parameter**, σ :

$$L_{ucsd}(h, y) = 1 - e^{-(y-h)^2 / \sigma^2}$$

← a constant that controls width of "bowl"



Scale parameter controls width of bowl



Empirical risk minimization

- ▶ We have salaries y_1, y_2, \dots, y_n .
- ▶ To find prediction, ERM says to minimize the average loss:

$$\begin{aligned} R_{ucsd}(h) &= \frac{1}{n} \sum_{i=1}^n L_{ucsd}(h, y_i) \\ &= \frac{1}{n} \sum_{i=1}^n \left[1 - e^{-(y_i - h)^2 / \sigma^2} \right] \end{aligned}$$

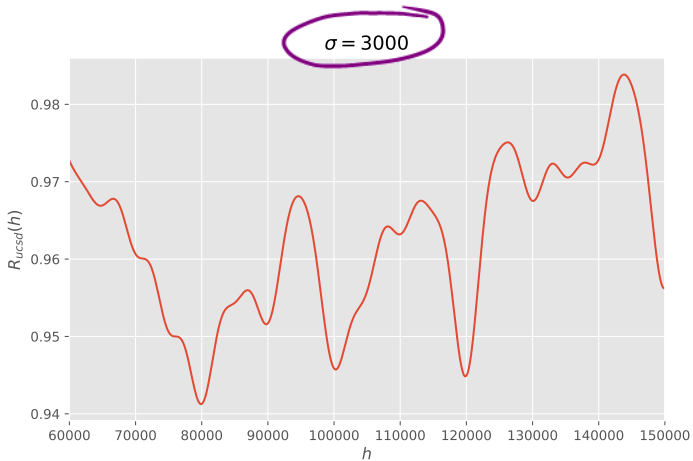
Let's plot R_{ucsd}

- ▶ Recall:

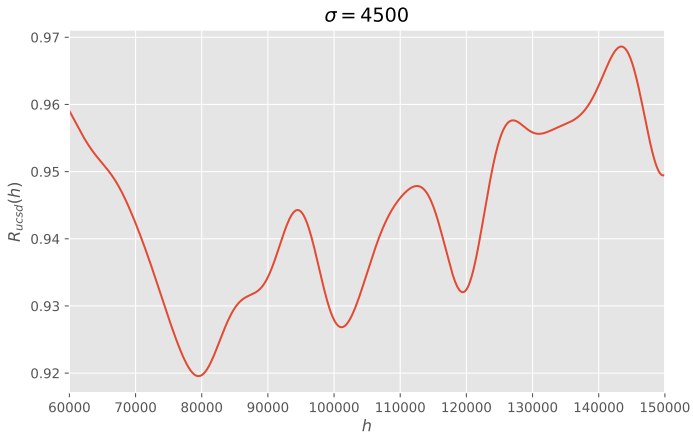
$$\underline{R_{ucsd}(h)} = \frac{1}{n} \sum_{i=1}^n \left[1 - e^{-(y_i - h)^2 / \sigma^2} \right]$$

- ▶ Once we have data y_1, y_2, \dots, y_n and a scale σ , we can plot $R_{ucsd}(h)$.
- ▶ Let's try several scales, σ , for the data scientist salary data.

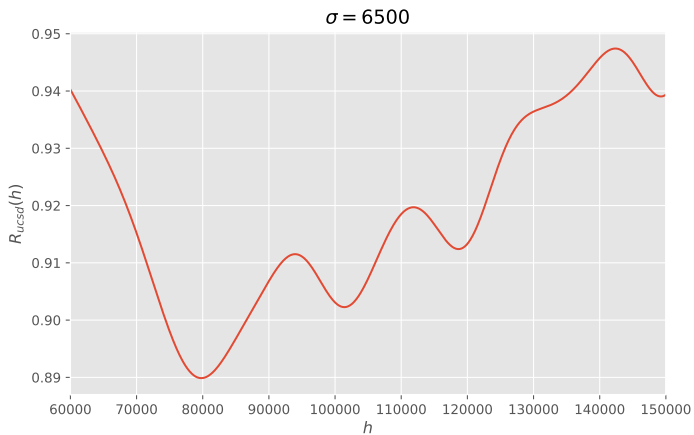
Plot of $R_{ucsd}(h)$



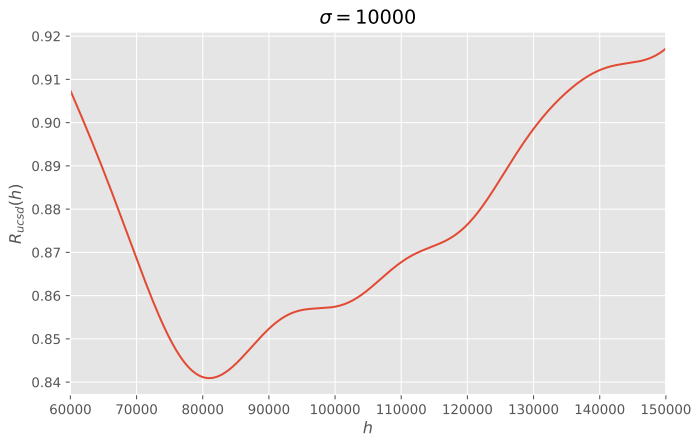
Plot of $R_{ucsd}(h)$



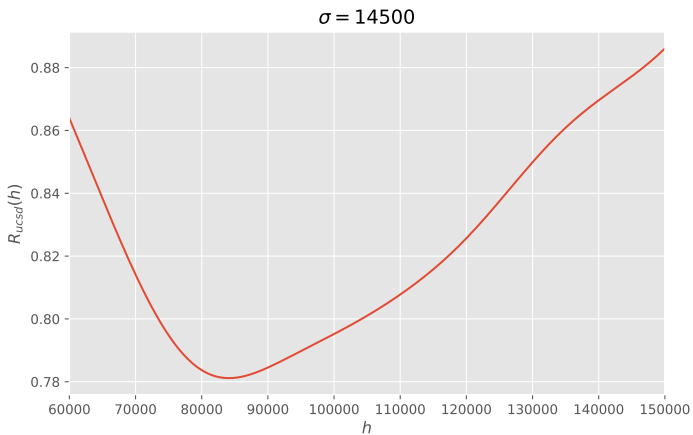
Plot of $R_{ucsd}(h)$



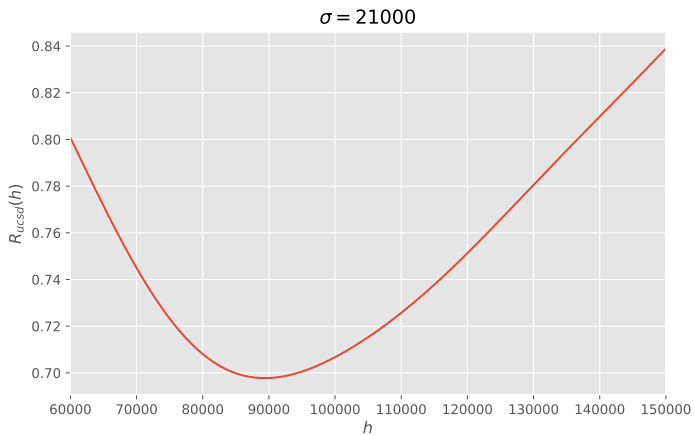
Plot of $R_{ucsd}(h)$



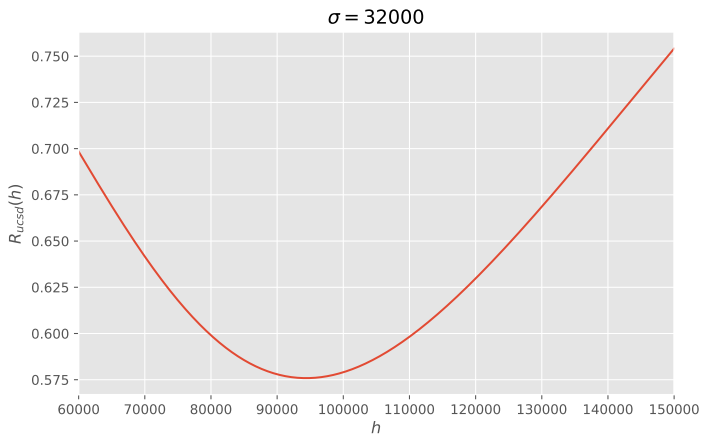
Plot of $R_{ucsd}(h)$



Plot of $R_{ucsd}(h)$



Plot of $R_{ucsd}(h)$



Minimizing R_{ucsd}

- ▶ To find the best prediction, we find h^* minimizing $R_{ucsd}(h)$.
- ▶ $R_{ucsd}(h)$ is **differentiable**.
- ▶ To minimize: take derivative, set to zero, solve.

Step 1: Taking the derivative

$$\frac{dR_{ucsd}}{dh} = \frac{d}{dh} \left(\frac{1}{r} \sum_{i=1}^n [1 - e^{-(y_i - h)^2 / \sigma^2}] \right)$$

chain rule

try it

Step 2: Setting to zero and solving

- ▶ We found:

$$\frac{d}{dh} R_{ucsd}(h) = \frac{2}{n\sigma^2} \sum_{i=1}^n (h - y_i) \cdot e^{-(h-y_i)^2/\sigma^2} = 0$$

- ▶ Now we just set to zero and solve for h :

$$\underline{x} \cdot e^x = 0$$

$$0 = \frac{2}{n\sigma^2} \sum_{i=1}^n (h - y_i) \cdot e^{-(h-y_i)^2/\sigma^2}$$

- ▶ We **can** calculate derivative, but we **can't** solve for h ; we're stuck again.

Summary

- ▶ Different loss functions lead to empirical risk functions that are minimized at various measures of **center**.
- ▶ The minimum values of these empirical risk functions are various measures of **spread**.
- ▶ We came up with a more complicated loss function, L_{ucsd} , that treats all outliers equally.
 - ▶ We weren't able to minimize its empirical risk R_{ucsd} by hand.
- ▶ **Next Time:** We'll learn a computational tool to approximate the minimizer of R_{ucsd} .