Lecture 9 – Regression in Action and Linear Algebra Review

DSC 40A, Spring 2023

Announcements

- Homework 3 is due **Tuesday at 11:59pm**.
  - Come to office hours. See `dsc40a.com/calendar` for the schedule.
  - It's a pretty long homework. Start early!
- Solutions to Groupwork 3 and Homework 2 are now available on Campuswire.
  - Reviewing them will help you on upcoming assignments and exams.

Agenda

- Recap of Lecture 8.
- Connection with correlation.
- Interpretation of formulas.
- Regression demo.
- Linear algebra review.

Recap of Lecture 8

The best linear prediction rule

- Last time, we used multivariable calculus to find the slope w_1^* and intercept w_0^* that minimized the MSE for a linear prediction rule of the form

$$H(x) = \overset{\text{int}}{w_0} + \overset{\text{slope}}{w_1} x$$

- In other words, we minimized this function:

$$R_{sq}(w_0, w_1) = \frac{1}{n} \sum_{i=1}^{n} \left(\underbrace{y_i}_{\text{actual}} - \underbrace{(w_0 + w_1 x_i)}_{\text{predicted}} \right)^2$$

Optimal parameters

- We found the optimal parameters to be:

$$w_1^* = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \qquad w_0^* = \bar{y} - w_1\bar{x}$$

- To make predictions about the future, we use the prediction rule

$$H^*(x) = w_0^* + w_1^* x$$

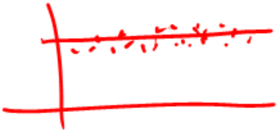
- This line is the **regression line**.

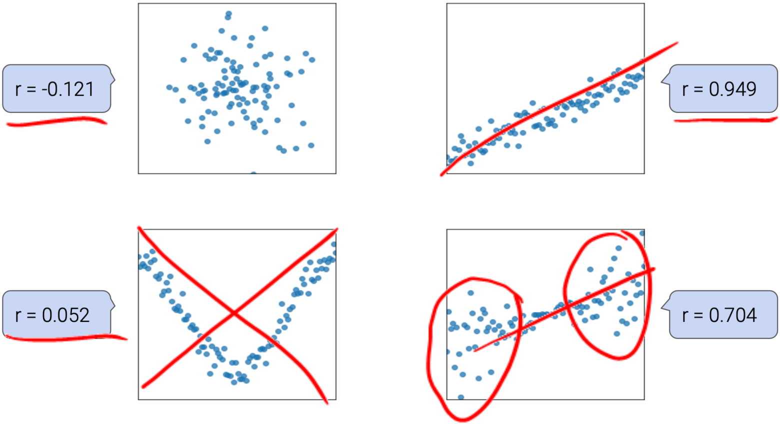
Connection with correlation

Correlation coefficient

- In DSC 10, you were introduced to the idea of correlation.
  - It is a measure of the strength of the **linear association** of two variables, *x* and *y*.
  - Intuitively, it measures how tightly clustered a scatter plot is around a straight line.
  - It ranges between -1 and 1.

Patterns in scatter plots





Definition of correlation coefficient

- The correlation coefficient, r, is defined as **the average of the product of x and y, when both are in standard units**.
  - Let σ_x be the standard deviation of the x_i's, and $\bar{x}$ be the mean of the x_i's.

1 SD

  - x_i in standard units is $\dfrac{x_i - \bar{x}}{\sigma_x}$. ← how many SDs above mean
  - The correlation coefficient is

$$\boxed{r = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{\sigma_x}\right)\cdot\left(\frac{y_i - \bar{y}}{\sigma_y}\right)}$$

Standard units

Another way to express w_1^*

- It turns out that w_1^*, the optimal slope for the linear prediction rule, can be written in terms of r!

$$\boxed{w_1^* = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}} = \boxed{r\frac{\sigma_y}{\sigma_x}} \leftarrow \text{formula from DSC 10}$$

- It's not surprising that r is related to w_1^*, since r is a measure of linear association.

- Concise way of writing w_0^* and w_1^*:

$$\boxed{w_1^* = r\frac{\sigma_y}{\sigma_x} \qquad w_0^* = \bar{y} - w_1^*\bar{x}}$$

Proof that $w_1^* = r\frac{\sigma_y}{\sigma_x}$

rhs:

$$r\frac{\sigma_y}{\sigma_x} = \left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i-\bar{x}}{\sigma_x}\right)\left(\frac{y_i-\bar{y}}{\sigma_y}\right)\right)\frac{\sigma_y}{\sigma_x}$$

distributive rule

$$= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i-\bar{x}}{\sigma_x}\right)\left(\frac{y_i-\bar{y}}{\cancel{\sigma_y}}\right)\left(\frac{\cancel{\sigma_y}}{\sigma_x}\right)$$

$$= \frac{1}{n}\cdot\frac{1}{(\sigma_x)^2}\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})$$

$$= \frac{\cancel{\frac{1}{n}}\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\cancel{\frac{1}{n}}\sum_{i=1}^{n}(x_i-\bar{x})^2} \leftarrow (\sigma_x)^2 = \mathrm{var}(x)$$

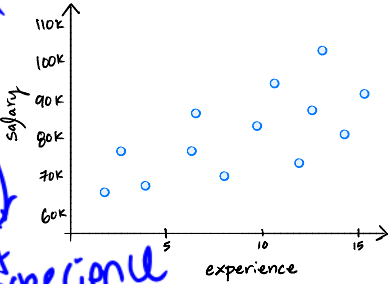
Interpretation of formulas

Interpreting the slope

units: \$ per year

represents how much more salary you'd make for every year of experience

$$w_1^* = r\frac{\sigma_y}{\sigma_x}$$

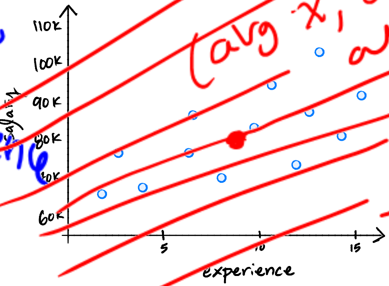


- σ_y and σ_x are always non-negative. As a result, the sign of the slope is determined by the sign of r.
- As the y values get more spread out, σ_y increases and so does the slope.
- As the x values get more spread out, σ_x increases and the slope decreases.

Interpreting the intercept

units: dollars
represents "new grad" salary - 0 years of experience

$$w_0^* = \bar{y} - w_1^* \bar{x}$$



(avg x, avg y) always falls on reg line

- What is $H^*(\bar{x})$?

$$H^*(x) = w_0^* + w_1^* \cdot x$$

$$H^*(\bar{x}) = w_0^* + w_1^* \bar{x}$$

$$= (\bar{y} - \cancel{w_1^* \bar{x}}) + \cancel{w_1^* \bar{x}}$$

$$= \bar{y} \leftarrow \text{avg } y$$

avg x

Discussion Question

We fit a linear prediction rule for salary given years of experience. Then everyone gets a $5,000 raise. Which of these happens?

$$w_0^* = \bar{y} - w_1^* \bar{x}$$

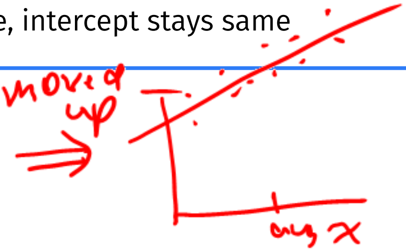
inc by 5000

a) slope increases, intercept increases

b) slope decreases, intercept increases

c) slope stays same, intercept increases

d) slope stays same, intercept stays same



Regression demo

Let’s see gradient descent in action. Follow along here.

Linear algebra review

Wait... why do we need linear algebra?

- Soon, we'll want to make predictions using more than one feature (e.g. predicting salary using years of experience and GPA).

- Thinking about linear regression in terms of **linear algebra** will allow us to find prediction rules that
  - use multiple features.

  - are non-linear.

- Before we dive in, let's review.

Matrices

rows first

columns

- An $m \times n$ **matrix** is a table of numbers with m rows and n columns.

- We use upper-case letters for matrices.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad 2 \times 3$$

- A^T denotes the transpose of A:

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad 3 \times 2$$

Matrix addition and scalar multiplication

- We can add two matrices only if they are the same size.
- Addition occurs elementwise:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 8 & 9 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 8 & 10 & 12 \\ 3 & 3 & 3 \end{bmatrix}$$

- Scalar multiplication occurs elementwise, too:

$$2 \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

Matrix-matrix multiplication

- We can multiply two matrices A and B only if

  # columns in A = # rows in B.

- If A is $m \times n$ and B is $n \times p$, the result is $m \times p$.
  - This is **very useful**.

- The ij entry of the product is:

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

2,1 2 1

=

3x3 3x2 3x2

Some matrix properties

- Multiplication is Distributive:

$$A(B + C) = AB + AC$$

- Multiplication is Associative:

$$(AB)C = A(BC)$$

- Multiplication is **not commutative**:

$$AB \neq BA$$

can't change order

- Transpose of sum:

$$(A + B)^T = A^T + B^T$$

- Transpose of product:

$$(AB)^T = \underline{B^T}\,\underline{A^T}$$

← order switches

Vectors

- An **vector** in $\mathbb{R}^n$ is an $n \times 1$ matrix.

(one-column matrix)

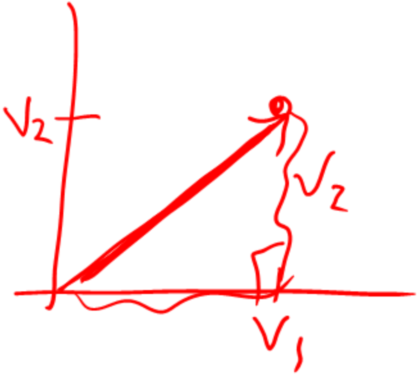
- We use lower-case letters for vectors.

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 5 \\ -3 \end{bmatrix}$$

- Vector addition and scalar multiplication occur elementwise.

Geometric meaning of vectors

- A vector $\vec{v} = (v_1, \dots, v_n)^T$ is an arrow to the point $(v_1, \dots, v_n)$ from the origin.



- The **length**, or **norm**, of $\vec{v}$ is $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$.

Dot products

- The **dot product** of two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ is denoted by:

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$$

- Definition:

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

multiply corresponding entries

add them all up

- The result is a **scalar**!

Discussion Question

Which of these is another expression for the length of $\vec{u}$?

a) $\vec{u} \cdot \vec{u}$

b) $\sqrt{\vec{u}^2}$

c) $\sqrt{\vec{u} \cdot \vec{u}}$

d) $\vec{u}^2$

don't make sense, can't square vector

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}_{n \times 1} \qquad \vec{u} \cdot \vec{u} = u_1^2 + u_2^2 + \dots + u_n^2$$

$$\vec{u} = \underbrace{\begin{bmatrix} \ \end{bmatrix} \quad n \times 1}$$

Properties of the dot product

- Commutative:

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} = \vec{u}^T \vec{v} = \vec{v}^T \vec{u}$$

- Distributive:

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

Matrix-vector multiplication

- Special case of matrix-matrix multiplication.
- Result is always a vector with same number of rows as the matrix.
- One view: a “mixture” of the columns.

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

- Another view: a dot product with the rows.

Discussion Question

If A is an $m \times n$ matrix and $\vec{v}$ is a vector in $\mathbb{R}^n$, what are the dimensions of the product $\vec{v}^T A^T A \vec{v}$?

a) $m \times n$ (matrix)
b) $n \times 1$ (vector)
c) 1×1 (scalar)
d) The product is undefined.

Summary

Summary, next time

- We can re-write the optimal parameters for the regression line

$$w_1^* = r\frac{\sigma_y}{\sigma_x} \qquad w_0^* = \bar{y} - w_1^*\bar{x}$$

- We can then make predictions using $H^*(x) = w_0^* + w_1^* x$.
- We will need linear algebra in order to generalize regression to work with multiple features.
- **Next time**: Continue linear algebra review. Formulate linear regression in terms of linear algebra.