Lecture 16 – Conditional Probability, Sequences and Permutations

DSC 40A, Spring 2023

Announcements

- Discussion is tonight at 7pm or 8pm in FAH 1101. Groupwork 6 is due **tonight at 11:59pm**.
- Homework 5 is released, due **Tuesday at 11:59pm**.
- **Important:** We've posted **many** probability resources on the resources tab of the course website. These will no doubt come in handy.
    - No more DSC 40A-specific readings, though the Probability Roadmap was written specifically for students of this course.

today's slides: on Campuswire

Agenda

- Conditional probability.
- Simpson’s Paradox.
- Sequences and permutations.

Conditional probability

Last time

1 complement rule (NOT)

- $\bar{A}$ is the complement of event A. $P(\bar{A}) = 1 - P(A)$.

- For any two events A and B

  2 addition rule (OR)

  $$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

  If A and B are mutually exclusive, this simplifies to

  no overlap

  $$P(A \cup B) = P(A) + P(B).$$

- The probability that events A and B both happen is

  3 multiplication rule (AND)

  $$P(A \cap B) = P(A)\underbrace{P(B|A)}_{\text{conditional}}.$$

- $P(B|A)$ is the conditional probability of B occurring, given that A occurs. If $P(B|A) =$ [redacted], then events A and B are independent.

  P(B)

Conditional probability

- The probability of an event may **change** if we have additional information about outcomes.
- Starting with the multiplication rule, $P(A \cap B) = P(A)P(B|A)$, we have that

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

assuming that $P(A) > 0$.

Example: pets

Suppose a family has two pets. Assume that it is equally likely that each pet is a dog or a cat. Consider the following two probabilities:

1. The probability that both pets are dogs given that **the oldest is a dog**.

2. The probability that both pets are dogs given that **at least one of them is a dog**.

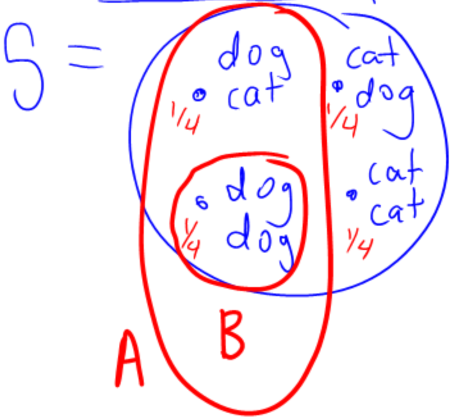
Discussion Question

Are these two probabilities equal?

a) Yes, they're equal
b) No, they're not equal

Example: pets

Let's compute the probability that both pets are dogs given that **the oldest is a dog**.



$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

$$= \frac{1/4}{1/2}$$

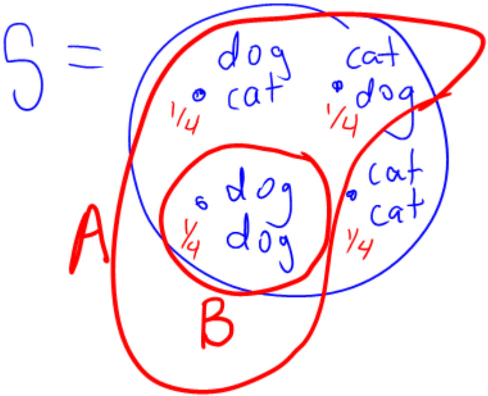
$$= 1/2$$

Example: pets

B

Let's now compute the probability that both pets are dogs given that **at least one of them is a dog**.

A



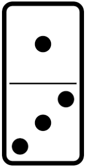
$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

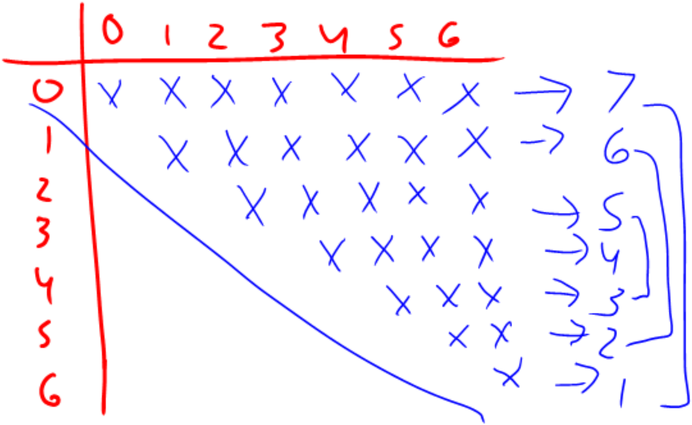
$$= \frac{1/4}{3/4}$$

$$= 1/3$$

Example: dominoes (source: 538)

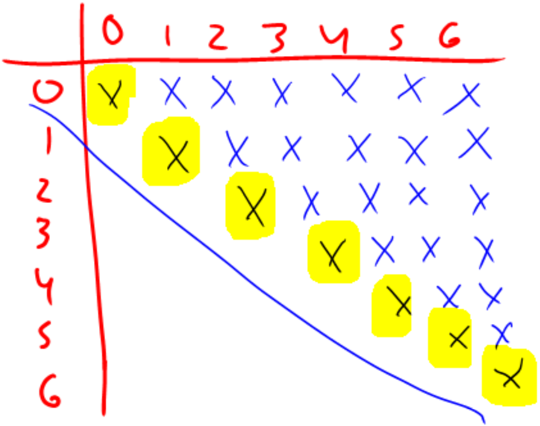
In a set of dominoes, each tile has two sides with a number of dots on each side: zero, one, two, three, four, five, or six. There are 28 total tiles, with each number of dots appearing alongside each other number (including itself) on a single tile.





Example: dominoes (source: 538)

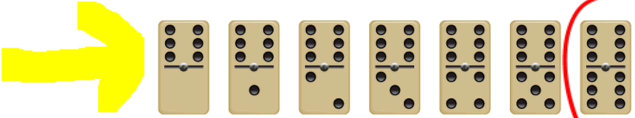
Question 1: What is the probability of drawing a “double” from a set of dominoes — that is, a tile with the same number on both sides?



$$\frac{7}{28} = \frac{1}{4}$$

Example: dominoes (source: 538)

Question 2: Now your friend picks a random tile from the set and tells you that at least one of the sides is a 6. What is the probability that your friend's tile is a double, with 6 on both sides?



$\frac{1}{7}$

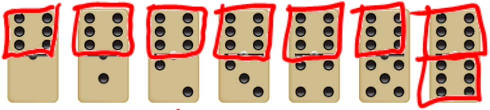
direct interpretation

	0	1	2	3	4	5	6
0	X	X	X	X	X	X	X
1		X	X	X	X	X	X
2			X	X	X	X	X
3				X	X	X	X
4					X	X	X
5						X	X
6							X

$$P(66 \mid \text{at least one } 6) = \frac{P(66 \text{ AND at least one } 6)}{P(\text{at least one } 6)} = \frac{1/28}{7/28} = 1/7$$

Example: dominoes (source: 538)

Question 3: Now you pick a random tile from the set and uncover only one side, revealing that it has six dots. What is the probability that this tile is a double, with six on both sides?



1/4

1/7

set of domino halves

$28 \cdot 2 = 56$ domino halves

8 of these domino halves have

direct interpretation: of 8 halves, 2 of them also have six dots across from it

$2/8 = 1/4 \rightarrow$

See 538's explanation here.

$$P\left(66 \,\middle|\, \begin{matrix}\text{uncovered} \\ \text{side had } 6\end{matrix}\right)$$

$$= \frac{P\left(66 \text{ and } \begin{matrix}\text{uncovered side} \\ \text{is } 6\end{matrix}\right)}{P(\text{uncovered side is } 6)}$$

$$= \frac{1/28}{8/56}$$

$\leftarrow$ of 28 dominoes, only 1 is $\boxed{\begin{matrix}6\\ \hline 6\end{matrix}}$

$\leftarrow$ of all 58 halves they could have uncovered, 8 have $\boxed{6}$

$$= 1/4$$

Simpson’s Paradox

Simpson's Paradox (source: nih.gov)

	Treatment A	Treatment B
Small kidney stones	81 successes / 87 (93%)	234 successes / 270 (87%)
Large kidney stones	192 successes / 263 (73%)	55 successes / 80 (69%)
Combined	273 successes / 350 (78%)	289 successes / 350 (83%)

Discussion Question

Which treatment is better?

a) Treatment A for all cases.
b) Treatment B for all cases.
c) Treatment A for small stones and B for large stones.
d) Treatment A for large stones and B for small stones.

Simpson's Paradox (source: nih.gov)

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Simpson's Paradox occurs when an association between two variables exists when the data is divided into subgroups, but reverses or disappears when the groups are combined.

- See more in DSC 80.

Sequences and permutations

Motivation

- Many problems in probability involve counting.
  - Suppose I flip a fair coin 100 times. What's the probability I see 34 heads?
  - Suppose I draw 3 cards from a 52 card deck. What's the probability they all are all from the same suit?
- In order to solve such problems, we first need to learn how to count. (# objects with certain properties)
- The area of math that deals with counting is called **combinatorics**.

Selecting elements (i.e. sampling)

- Many experiments involve choosing k elements randomly from a group of n possible elements. This group is called a **population**.
  - If drawing cards from a deck, the population is the deck of all cards.
  - If selecting people from DSC 40A, the population is everyone in DSC 40A.
- Two decisions:
  - Do we select elements with or without **replacement**?
  - Does the **order** in which things are selected matter?

Sequences

- A **sequence** of length k is obtained by selecting k elements from a group of n possible elements **with replacement**, such that **order matters**.

- **Example:** Draw a card (from a standard 52-card deck), put it back in the deck, and repeat 4 times.

ex.) J♥, Q♥, 3♠, 4♦

$$52 \cdot 52 \cdot 52 \cdot 52 = 52^4$$

- **Example:** A UCSD PID starts with "A" then has 8 digits. How many UCSD PIDs are possible?

ex.) A12345678

$$1 \cdot 10 \cdot 10 \cdot \ldots \cdot 10 = 10^8$$

Sequences

In general, the number of ways to select k elements from a group of n possible elements such that **repetition is allowed** and **order matters** is n^k.

$$\underbrace{n \cdot n \cdot n \cdot \ldots \cdot n}_{k \text{ terms}}$$

(Note: We mentioned this fact in the lecture on clustering!)

Permutations

- A **permutation** is obtained by selecting *k* elements from a group of *n* possible elements **without replacement**, such that **order matters**.

- **Example:** Draw 4 cards (without replacement) from a standard 52-card deck.

ex.) 7♣, 10♥, J♠, 3♦ $\quad 52-4+1$

$$52 \cdot 51 \cdot 50 \cdot \boxed{49}$$

- **Example:** How many ways are there to select a president, vice president, and secretary from a group of 8 people?

number the people: 1,2,3,4,5,6,7,8

ex.) pres: 5, vp: 8, secretary: 2

$$8 \cdot 7 \cdot 6$$

Permutations

- In general, the number of ways to select k elements from a group of n possible elements such that **repetition is not allowed** and **order matters** is

$$P(n, k) = (n)(n-1)\dots(n-k+1)$$

- To simplify: recall that the definition of $n!$ is

$$n! = (n)(n-1)\dots(2)(1)$$

- Given this, we can write

$$P(n, k) = \frac{n!}{(n-k)!}$$

Discussion Question

UCSD has 7 colleges. How many ways can I rank my top 3 choices?

a) 21
b) 210
c) 343
d) 2187
e) None of the above.

Special case of permutations

- Suppose we have n people. The total number of ways I can rearrange these n people in a line is

- This is consistent with the formula

$$P(n, n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

Summary, next time

Summary

- The **conditional probability** of B given A is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

- A **sequence** is obtained by selecting k elements from a group of n possible elements with replacement, such that order matters.
  - Number of sequences: n^k.

- A **permutation** is obtained by selecting k elements from a group of n possible elements without replacement, such that order matters.
  - Number of permutations: $P(n, k) = \frac{n!}{(n-k)!}$.

- **Next time: combinations**, where order doesn't matter.