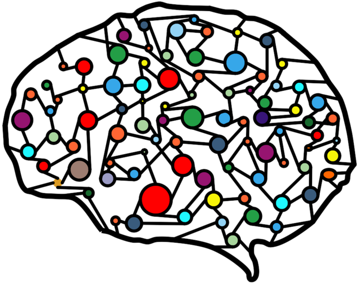
Lecture 18 - Probabability and Combinatorics Examples



DSC 40A, Spring 2023

Announcements

- Homework 5 is due **tomorrow at 11:59pm**.
- RSVP for the DSC Undergrad Town Hall tomorrow from 1:30-3:30pm in the SDSC Auditorium.
    - A chance to talk about what's going on in the department, raise concerns, talk to professors, etc.

Agenda

- Review of combinatorics.
- Lots of examples.

Review of combinatorics

Combinatorics as a tool for probability

- If S is a sample space consisting of equally-likely outcomes, and A is an event, then $P(A) = \frac{|A|}{|S|}$.

- In many examples, this will boil down to using permutations and/or combinations to count $|A|$ and $|S|$.

- **Tip:** Before starting a probability problem, always think about what the sample space S is!

Sequences

- A **sequence** of length k is obtained by selecting k elements from a group of n possible elements **with replacement**, such that **order matters**.

- **Example:** You roll a die 10 times. How many different sequences of results are possible?

3 (1st) 2 (2nd) . . . 6 (10th)

$(6)^{10}$

Sequences

In general, the number of ways to select k elements from a group of n possible elements such that **repetition is allowed** and **order matters** is

$$n^k.$$

Permutations

- A **permutation** is obtained by selecting k elements from a group of n possible elements **without replacement**, such that **order matters**.

- **Example:** How many ways are there to select a president, vice president, and secretary from a group of 8 people?

$$\underset{\text{pres}}{\underline{8}} \cdot \underset{\text{vp}}{\underline{7}} \cdot \underset{\text{sec}}{\underline{6}}$$

$$= P(8,3) = \frac{8!}{5!} = 8 \cdot 7 \cdot 6$$

Permutations

- In general, the number of ways to select k elements from a group of n possible elements such that **repetition is not allowed** and **order matters** is

$$\begin{aligned} P(n,k) &= (n)(n-1)\ldots(n-k+1) \\ &= \frac{n!}{(n-k)!} \end{aligned}$$

Combinations

[- which elements are included]

- A **combination** is a set of k items selected from a group of n possible elements **without replacement**, such that **order does not matter**.

- **Example:** How many ways are there to select a committee of 3 people from a group of 8 people?

$$C(8,3) = \frac{P(8,3)}{3!}$$

$\underset{\text{pres}}{8} \cdot \underset{\text{vp}}{7} \cdot \underset{\text{sec}}{6}$ except ABC is same as BAC, CAB, CBA, BAC, BCA

Combinations

In general, the number of ways to select k elements from a group of n elements such that **repetition is not allowed** and **order does not matter** is

$$C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!k!}$$

\# ways to choose a set of k things from n

The symbol $\binom{n}{k}$ is pronounced "n choose k", and is also known as the **binomial coefficient**.

Lots of examples

order matters

	Yes	No
with repl.	Sequence	dominoes
without repl.	Permutation	combination

dominoes → sums of combinations

Discussion Question

A domino consists of two faces, each with anywhere between 0 and 6 dots. A set of dominoes consists of every possible combination of dots on each face.

How many dominoes are in the set of dominoes?

a) $\binom{7}{2}$ = C(7,2) → doesn't include doubles

b) $\binom{7}{1} + \binom{7}{2}$ = C(7,1) + C(7,2)

c) $P(7,2)$ → counts [1|4] ≠ [4|1]

d) $\frac{P(7,2)}{P(7,1)} 7!$

7 options: 0, 1, 2, 3, 4, 5, 6

C(7,2)

doubles not counted by C(7,2)

doubles counted by C(7,1)

0, 1, 2, 3, 4, 5, 6

C(7,1) = 7

order matters? NO

with/without replacement? WITH (can have doubles)

Selecting students — overview

We’re going answer the same question using several different techniques.

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

each person equally likely to be picked

can't be picked multple times

Selecting students (Method 1: using permutations)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

S = all possible permutations (ordered arrangements) of 5 students chosen from 20 (label students A, B, C, ..., T) ← Avi is A

ex) RCAKL ← because all outcomes are equally likely, we can calculate prob. as

$$\frac{\text{\# perms including A}}{\text{total \# of perms}}$$

denom:

how many perms possible?

$\underset{20}{\underline{R}} \cdot \underset{19}{\underline{C}} \cdot \underset{18}{\underline{A}} \cdot \underset{17}{\underline{K}} \cdot \underset{16}{\underline{L}} = P(20,5) = \frac{20!}{15!}$

numerator?

how many perms include A (for Avi)?

$\underset{1}{\underline{A}}\ \underset{19}{\underline{M}}\ \underset{18}{\underline{N}}\ \underset{17}{\underline{D}}\ \underset{16}{\underline{B}} \rightarrow 19 \cdot 18 \cdot 17 \cdot 16$

counts perms where A is 1st

how many where A is 2nd?

$\underset{19}{\underline{C}}\ \underset{1}{\underline{A}}\ \underset{18}{\underline{N}}\ \underset{17}{\underline{S}}\ \underset{16}{\underline{T}} \rightarrow$ another $19 \cdot 18 \cdot 17 \cdot 16$

answer: $5 \cdot 19 \cdot 18 \cdot 17 \cdot 16 = 5 \cdot P(19,4) = 5 \cdot \frac{19!}{15!}$

$$\frac{\text{num}}{\text{denom}} = \frac{5 \cdot \frac{19!}{15!}}{\frac{20!}{15!}}$$

$$= 5 \cdot \frac{19!}{\cancel{15!}} * \frac{\cancel{15!}}{20!}$$

$$= 5 \cdot \frac{1}{20}$$

$$= 1/4$$

Selecting students (Method 2: using permutations and the complement)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

S = perms

$$\frac{\#\text{perms including A}}{\text{total \# perms}} = \frac{\text{total \# perms} - \#\text{perms not including A}}{P(20,5)}$$

$$= \frac{P(20,5) - \#\text{perms not including A}}{P(20,5)}$$

$$= \frac{P(20,5) - P(19,5)}{P(20,5)} = 1/4$$

how many perms don't include A?

$$\underset{19}{\underline{S}} \cdot \underset{18}{\underline{I}} \cdot \underset{17}{\underline{J}} \cdot \underset{16}{\underline{R}} \cdot \underset{15}{\underline{E}} = P(19,5) = \frac{19!}{14!}$$

Selecting students (Method 3: using combinations)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

S = set of 5 selected students (don't care about order)

ex.) $\{C, D, J, K, P\}$

$$\frac{\#\text{ sets including A}}{\#\text{ sets}}$$

num / denom

how many sets?

$$C(20,5) = \binom{20}{5} = \frac{20!}{15!\,5!}$$

how many sets include A?

$$C(19,4) = \binom{19}{4} = \frac{19!}{15!\,4!}$$

Selecting students (Method 3: using combinations)

Question 1, Part 1 (Denominator): If you draw a sample of size 5 at random without replacement from a population of size 20, how many different **sets** of individuals could you draw?

how many sets?

$$C(20,5) = \binom{20}{5} = \frac{20!}{15!\,5!}$$

Selecting students (Method 3: using combinations)

Question 1, Part 2 (Numerator): If you draw a sample of size 5 at random without replacement from a population of size 20, how many different **sets** of individuals include Avi?

how many sets include A?

$$C(19,4) = \binom{19}{4} = \frac{19!}{15!\,4!}$$

among 19 other students

select 4 to go with Avi

Selecting students (Method 3: using combinations)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

$$\frac{C(19,4)}{C(20,5)} = \frac{\frac{19!}{15!\,4!}}{\left(\frac{20!}{15!\,5!}\right)} = \frac{19!}{15!\,4!} \cdot \frac{15!\,5!}{20!} = \frac{5}{20} = \frac{1}{4}$$

Selecting students (Method 4: "the easy way")

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

notice: answer is $\frac{5}{20} = \frac{1}{4}$

think of selecting 5 students as follows:
randomize all 20 students, reorder them
line up 20 students in random order + take first five

S = positions where Avi can go $\frac{\#\text{ good positions}}{\#\text{ positions}} = \frac{5}{20}$

With vs. without replacement

Discussion Question

We've determined that a probability that a random sample of 5 students from a class of 20 **without replacement** contains Avi (one student in particular) is $\frac{1}{4}$.

Suppose we instead sampled **with replacement**. Would the resulting probability be equal to, greater than, or less than $\frac{1}{4}$?

a) Equal to
b) Greater than
c) Less than

Art supplies

Question 2, Part 1: We have 12 art supplies: 5 markers and 7 crayons. In how many ways can we select 4 art supplies?

Art supplies

Question 2, Part 2: We have 12 art supplies: 5 markers and 7 crayons. In how many ways can we select 4 art supplies such that we have...

1. 2 markers and 2 crayons?
2. 3 markers and 1 crayon?
3. At least 2 markers?

Art supplies

Question 2, Part 3: We have 12 art supplies: 5 markers and 7 crayons. We randomly select 4 art supplies. What's the probability that we selected at least 2 markers?

Fair coin

Question 3: Suppose we flip a fair coin 10 times. What is the probability that we see an equal number of heads and tails?

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

Unfair coin

Question 4: Suppose we flip a coin **that is not fair**, but instead has $P(\text{heads}) = \frac{1}{3}$, 10 times. Assume that each flip is independent.

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

Summary

Summary

- A **sequence** is obtained by selecting k elements from a group of n possible elements with replacement, such that order matters.
  - Number of sequences: n^k.

- A **permutation** is obtained by selecting k elements from a group of n possible elements without replacement, such that order matters.
  - Number of permutations: $P(n,k) = \frac{n!}{(n-k)!}$.

- A **combination** is obtained by selecting k elements from a group of n possible elements without replacement, such that order does not matter.
  - Number of combinations: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.