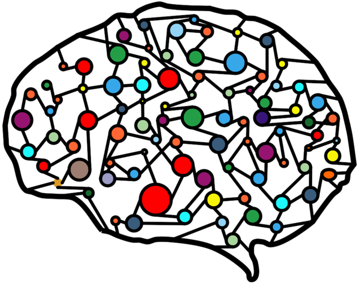
# Lecture 19 – More Probabability and Combinatorics Examples



DSC 40A, Spring 2023

## Announcements

- Discussion is tonight at 7pm or 8pm in FAH 1101.
  - Come to work on Groupwork 6, which is due **tonight at 11:59pm**.
- Homework 6 is released, due **Tuesday at 11:59pm**.
- Don't forget to read through the solutions to past assignments before doing the next assignment. This is especially useful for probability and combinatorics to learn new ways of solving problems.
  - See the pinned post on Campuswire.

# Agenda

- Lots of examples.

# Last time

Last time we answered the same question using several different techniques.

**Question 1:** There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

$1/4$

# With vs. without replacement

always get 5 people

## Discussion Question

We've determined that a probability that a random sample of 5 students from a class of 20 **without replacement** contains Avi (one student in particular) is $\frac{1}{4}$.

Suppose we instead sampled **with replacement**. Would the resulting probability be equal to, greater than, or less than $\frac{1}{4}$?

a) Equal to
b) Greater than
c) Less than

may not get 5 unique people, may just get same person repeated

get $\leq 5$ people

another way:

- without replacement

$P(\text{Avi on 1st pick}) = \frac{1}{20}$

$P(\text{Avi on 2nd pick} \mid \text{didn't get Avi on 1st pick}) = \frac{1}{19}$

- with replacement

$P(\text{Avi on 1st pick}) = \frac{1}{20}$

$P(\text{Avi on 2nd pick} \mid \cancel{\text{didn't get Avi on 1st pick}}) = \frac{1}{20}$

irrelevant (independent)

extreme: randomly sample 20 people from 20 -

without replacement

$P(\text{Avi}) = 1$

with replacement

$P(\text{Avi}) < 1$

# Art supplies

**Question 2, Part 1:** We have 12 art supplies: 5 markers and 7 crayons. In how many ways can we select 4 art supplies?

$$C(12,4) = \binom{12}{4}$$

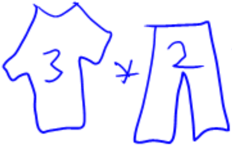
context: order doesn't matter

# Art supplies

**Question 2, Part 2:** We have 12 art supplies: 5 markers and 7 crayons. In how many ways can we select 4 art supplies such that we have...

1. 2 markers and 2 crayons?
2. 3 markers and 1 crayon?

$C(5,2) * C(7,2)$




$= 6$ outfits

$C(5,3) * \underbrace{C(7,1)}_{7}$

choose which 2 to take

$\frac{5!}{2!\,3!}$

choose which 3 to not take

$C(5,2) = C(5,3)$

$C(n,k) = C(n,n-k)$

Another thing to know!

# Art supplies

**Question 2, Part 3:** We have 12 art supplies: 5 markers and 7 crayons. We randomly select 4 art supplies. What's the probability that we selected at least 2 markers?

$S$ = all sets of 4 art supplies

$|S| = C(12,4)$

are all elements of $S$ equally likely? yes

$$P(\text{at least 2 markers}) = \frac{\#\text{ sets of 4 art supplies that include at least 2 markers}}{\#\text{ sets of 4 art supplies}}$$

| | |
|---|---|
| 0 markers | $\binom{5}{0}\binom{7}{4}$ |
| 1 marker | $\binom{5}{1}\binom{7}{3}$ |
| 2 markers | $\binom{5}{2}\binom{7}{2}$ |
| 3 markers | $\binom{5}{3}\binom{7}{1}$ |
| 4 markers | $\binom{5}{4}\binom{7}{0}$ |

total is $|S|$

$$= \frac{\binom{5}{2}\binom{7}{2}+\binom{5}{3}\binom{7}{1}+\binom{5}{4}\binom{7}{0}}{\binom{12}{4}} = \frac{\binom{12}{4}-\left(\binom{5}{0}\binom{7}{4}+\binom{5}{1}\binom{7}{3}\right)}{\binom{12}{4}}$$

# Fair coin

**Question 3:** Suppose we flip a **fair coin** 10 times.

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

$$\left(\frac{1}{2}\right)^{10} = \frac{1}{2^{10}}$$

$$\frac{\#\text{ seq. with } 5H, 5T}{\#\text{ seq}} = \frac{C(10,5)}{2^{10}}$$

← of 10 positions

H H T H H T T H T T

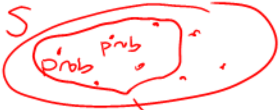
select 5 of them for tails

# Unfair coin

**Question 4:** Suppose we flip an **unfair coin** 10 times. The coin is biased such that for each flip, $\boxed{P(\text{heads}) = \frac{1}{3}}$.

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

$$\cancel{\frac{\#\text{"good" seq}}{\text{total \# seq}} = \frac{1}{2^{10}}}$$




add up probs for each outcome in event

not all outcomes are equally likely

$$\underset{\frac{2}{3}}{T}\ \underset{\frac{1}{3}}{H}\cdot\underset{\frac{2}{3}}{T}\ \underset{\frac{2}{3}}{T}\ \underset{\frac{1}{3}}{H}\ T\ H\ H\ T\ H \quad \cdots \quad \left(\frac{2}{3}\right)^5\cdot\left(\frac{1}{3}\right)^5$$

$P(H) = 1/3$ flip 10 times

Prob of equal # of H and T

$S$ = all sequences of H's + T's

- THTTHTHHTH prob: $\left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^5$
- prob: $\left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^5$
- HHHHHHHHHH prob: $\left(\frac{1}{3}\right)^{10}$

event I care about is equal # of H and T (five of each)

total prob of event E is $\sum_{s \in E} \text{prob}(s)$

$$\sum_{s \in E} \left(\frac{2}{3}\right)^5 \cdot \left(\frac{1}{3}\right)^5$$

$$= \left(\begin{matrix}\#\text{ elements} \\ \text{in } E\end{matrix}\right) * \left(\frac{2}{3}\right)^5 * \left(\frac{1}{3}\right)^5$$

$$= C(10,5) * \left(\frac{2}{3}\right)^5 * \left(\frac{1}{3}\right)^5$$

# Deck of cards

- There are 52 cards in a standard deck (4 suits, 13 values).

4 suits 13 values

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- In poker, each player is dealt 5 cards, called a **hand**. The order of cards in a hand does not matter.

Set

# Deck of cards

1. How many 5 card hands are there in poker?

$$C(52,5)$$

2. How many 5 card hands are there where all cards are of the same suit (a **flush**)?

$$4 \cdot \binom{13}{5}$$

$$\frac{52 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5!}$$

ex.) $\underset{52}{3\diamondsuit}, \underset{12}{10\diamondsuit}, \underset{11}{A\diamondsuit}, \underset{10}{4\diamondsuit}, \underset{9}{2\diamondsuit}$

what suit? 4 options

what 5 values? $C(13,5)$

52 options for 1st card

*12 options for 2nd card ...

3. How many 5 card hands are there that include a **four-of-a-kind** (four cards of the same value)?

$13 \cdot 48$

$12 \cdot 4$

ex.) 5♡, 5◇, 5♤, 5♧, 10♡

what # to be repeated? 13 values

what other card? $52-4=48$ other cards

4. How many 5 card hands are there that have a **straight** (all card values consecutive)?

need: which values? start 2, 3, …, 10

9 options

7, 8, 9, 10, J

$9 \cdot 4, 4, 4 \cdot 4, 4 = \boxed{9 \cdot 4^5}$

4 options for suit

5. How many 5 card hands are there that are a **straight flush** (all card values consecutive and of the same suit)?

choose values: 9 options

ex.) 9, 10, J, Q, K

choose suits: only choose one suit

4 options

$\boxed{9 \cdot 4}$

6. How many 5 card hands are there that include exactly **one pair** (values aabcd)?

ex.) K♢, K♠, 3♡, 5♣, 7♢

What value to repeat? 13 options

which pair of suits? $\binom{4}{2} = C(4,2)$ options

for other cards:

which values? $\binom{12}{3} = C(12,3)$ options

which suits? $4 \cdot 4 \cdot 4 = 4^3$ options

$$\boxed{13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3}$$

## Summary

# Summary

- A **sequence** is obtained by selecting $k$ elements from a group of $n$ possible elements with replacement, such that order matters.
  - Number of sequences: $n^k$.

- A **permutation** is obtained by selecting $k$ elements from a group of $n$ possible elements without replacement, such that order matters.
  - Number of permutations: $P(n, k) = \frac{n!}{(n-k)!}$.

- A **combination** is obtained by selecting $k$ elements from a group of $n$ possible elements without replacement, such that order does not matter.
  - Number of combinations: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.