Module 16 – Conditional Probability, Sequences and Permutations

DSC 40A, Summer 2023

Agenda

- Conditional probability.
- Simpson's Paradox.
- Sequences and permutations.

Conditional probability

Last module

- $\bar{A}$ is the complement of event A. $P(\bar{A}) = 1 - P(A)$.
- For any two events A and B

  $$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

  If A and B are mutually exclusive, this simplifies to

  $$P(A \cup B) = P(A) + P(B).$$

- The probability that events A and B both happen is

  $$P(A \cap B) = P(A)P(B|A).$$

- $P(B|A)$ is the conditional probability of B occurring, given that A occurs. If $P(B|A) = P(B)$, then events A and B are independent.

Conditional probability

- The probability of an event may **change** if we have additional information about outcomes.

- Starting with the multiplication rule, $P(A \cap B) = P(A)P(B|A)$, we have that

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

  assuming that $P(A) > 0$.

Example: pets

Suppose a family has two pets. Assume that it is equally likely that each pet is a dog or a cat. Consider the following two probabilities:

1. The probability that both pets are dogs "given that" **the oldest is a dog**.

2. The probability that both pets are dogs given that **at least one of them is a dog**. "oldest is a dog" vs "I have a dog"

Discussion Question

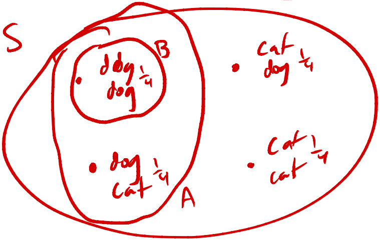
Are these two probabilities equal?

a) Yes, they're equal
b) No, they're not equal

Example: pets

B

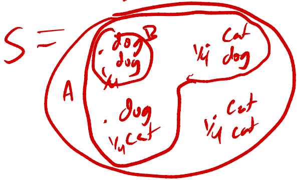
Let's compute the probability that both pets are dogs given that **the oldest is a dog**. ← A



$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

Example: pets

Let's now compute the probability that both pets are dogs given that **at least one of them is a dog**.

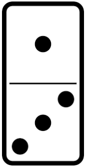


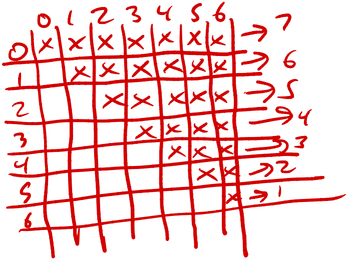
$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$$

$$\frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Example: dominoes (source: 538)

In a set of dominoes, each tile has two sides with a number of dots on each side: zero, one, two, three, four, five, or six. There are 28 total tiles, with each number of dots appearing alongside each other number (including itself) on a single tile.





Example: dominoes (source: 538)

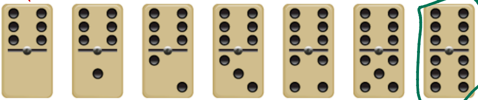
Question 1: What is the probability of drawing a “double” from a set of dominoes — that is, a tile with the same number on both sides?

$7/28 = 1/4$

Example: dominoes (source: 538)

Question 2: Now your friend picks a random tile from the set and tells you that at least one of the sides is a 6. What is the probability that your friend's tile is a double, with 6 on both sides?

$\geq \frac{1}{6}$



$\frac{1}{7}$

A is at least one 6
B is double 6

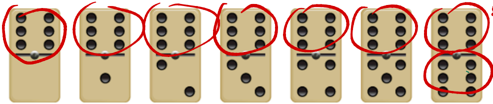
n=28

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{\frac{1}{28}}{\frac{7}{28}} = \frac{1}{7}$$

Example: dominoes (source: 538)

Question 3: Now you pick a random tile from the set and uncover only one side, revealing that it has six dots. What is the probability that this tile is a double, with six on both sides?

8 total



has 6 across from it

has 6 across from it

$\frac{1}{4}$

$2 \cdot 28 = 56$ domino halves

8 have a

$$P(66 \mid \text{uncovered side is a } 6) = \frac{P(66 \text{ and uncovered is } 6)}{P(\text{uncovered side is } 6)} = \frac{2/56}{8/56} = \boxed{\frac{1}{4}}$$

See 538's explanation here.

Simpson's Paradox

Simpson's Paradox (source: nih.gov)

harder cases

	Treatment A	Treatment B
Small kidney stones	81 successes / 87 (93%)	234 successes / 270 (87%)
Large kidney stones	192 successes / 263 (73%)	55 successes / 80 (69%)
Combined	273 successes / 350 (78%)	289 successes / 350 (83%)

Discussion Question

Which treatment is better?

a) Treatment A for all cases.
b) Treatment B for all cases.
c) Treatment A for small stones and B for large stones.
d) Treatment A for large stones and B for small stones.

Simpson's Paradox (source: nih.gov)

	Treatment A	Treatment B
Small kidney stones	81 successes / 87 (93%)	234 successes / 270 (87%)
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Simpson's Paradox occurs when an association between two variables exists when the data is divided into subgroups, but reverses or disappears when the groups are combined.

- See more in DSC 80.

Sequences and permutations

Motivation

- Many problems in probability involve counting.
  - Suppose I flip a fair coin 100 times. What's the probability I see 34 heads?
  - Suppose I draw 3 cards from a 52 card deck. What's the probability they all are all from the same suit?
- In order to solve such problems, we first need to learn how to count.
- The area of math that deals with counting is called **combinatorics**.

Selecting elements (i.e. sampling)

- Many experiments involve choosing k elements randomly from a group of n possible elements. This group is called a **population**.
  - If drawing cards from a deck, the population is the deck of all cards.
  - If selecting people from DSC 40A, the population is everyone in DSC 40A.
- Two decisions:
  - Do we select elements with or without **replacement**?
  - Does the **order** in which things are selected matter?

Sequences

- A **sequence** of length k is obtained by selecting k elements from a group of n possible elements **with replacement**, such that **order matters**.

- **Example:** Draw a card (from a standard 52-card deck), put it back in the deck, and repeat 4 times. → assume order matters

  ex) 2♡, 3♦, K♠, Q♣

  with replacement → 52 52 52 52 — 52^4 distinct sequences

  w/o replacement 52 51 50 49 — $52 \cdot 51 \cdot 50 \cdot 49$ distinct sequences

- **Example:** A UCSD PID starts with "A" then has 8 digits. How many UCSD PIDs are possible?

  A 1 2 3 4 5 6 7 8

  1 10 10 10 10 10 10 10 10 $= 10^8$ options

Sequences

"with replacement"

In general, the number of ways to select k elements from a group of n possible elements such that **repetition is allowed** and **order matters** is n^k.

$$\underbrace{n \cdot n \cdot n \cdots n}_{k \text{ terms}}$$

(Note: We mentioned this fact in the lecture on clustering!)

Permutations

- A **permutation** is obtained by selecting k elements from a group of n possible elements **without replacement**, such that **order matters**.

- **Example:** Draw 4 cards (without replacement) from a standard 52-card deck.

- **Example:** How many ways are there to select a president, vice president, and secretary from a group of $\underbrace{\text{8 people}}_{1,2,3,4,5,6,7,8}$?

pres vp sec

$8 \cdot 7 \cdot 6$

Permutations

- In general, the number of ways to select k elements from a group of n possible elements such that **repetition is not allowed** and **order matters** is

$$P(n, k) = (n)(n-1)...(n-k+1)$$

- To simplify: recall that the definition of $n!$ is

$$n! = (n)(n-1)...(2)(1)$$

- Given this, we can write

$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$$P(n, k) = \frac{n!}{(n-k)!}$$

Summary, next module

Summary

- The **conditional probability** of B given A is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

- A **sequence** is obtained by selecting k elements from a group of n possible elements with replacement, such that order matters.
  - Number of sequences: n^k.

- A **permutation** is obtained by selecting k elements from a group of n possible elements without replacement, such that order matters.
  - Number of permutations: $P(n,k) = \frac{n!}{(n-k)!}$.

- **Next module:** **combinations**, where order doesn't matter.