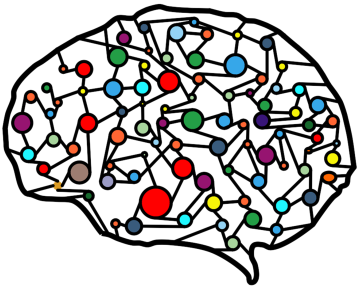
Module 17 - Sequences, Permutations, and Combinations



DSC 40A, Summer 2023

Agenda

- Sequences, permutations, and combinations.

Motivation

- Many problems in probability involve counting.
  - Suppose I flip a fair coin 100 times. What's the probability I see 34 heads?
  - Suppose I draw 3 cards from a 52 card deck. What's the probability they all are all from the same suit?
- In order to solve such problems, we first need to learn how to count.
- The area of math that deals with counting is called **combinatorics**.

Selecting elements (i.e. sampling)

- Many experiments involve choosing k elements randomly from a group of n possible elements. This group is called a **population**.
  - If drawing cards from a deck, the population is the deck of all cards.
  - If selecting people from DSC 40A, the population is everyone in DSC 40A.
- Two decisions:
  - Do we select elements with or without **replacement**?
  - Does the **order** in which things are selected matter?

Sequences

- A **sequence** of length k is obtained by selecting k elements from a group of n possible elements **with replacement**, such that **order matters**.

- **Example:** Draw a card (from a standard 52-card deck), put it back in the deck, and repeat 4 times.

  52 52 52 52

- **Example:** A UCSD PID starts with "A" then has 8 digits. How many UCSD PIDs are possible?

Sequences

In general, the number of ways to select k elements from a group of n possible elements such that **repetition is allowed** and **order matters** is n^k.

(Note: We mentioned this fact in the lecture on clustering!)

Permutations

- A **permutation** is obtained by selecting k elements from a group of n possible elements **without replacement**, such that **order matters**.

- **Example:** Draw 4 cards (without replacement) from a standard 52-card deck.

- **Example:** How many ways are there to select a president, vice president, and secretary from a group of 8 people?

Permutations

- In general, the number of ways to select k elements from a group of n possible elements such that **repetition is not allowed** and **order matters** is

$$P(n, k) = (n)(n-1)\ldots(n-k+1)$$

- To simplify: recall that the definition of $n!$ is

$$n! = (n)(n-1)\ldots(2)(1)$$

- Given this, we can write

$$P(n, k) = \frac{n!}{(n-k)!}$$

Discussion Question

UCSD has 7 colleges. How many ways can I rank my top 3 choices?

order matters

a) 21
b) 210
c) 343
d) 2187
e) None of the above.

$$P(7,3) = \frac{n!}{(n-k)!} = \frac{7!}{4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$= 210$$

Special case of permutations

- Suppose we have n people. The total number of ways I can rearrange these n people in a line is

- This is consistent with the formula

$$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

Combinations

- A **combination** is a set of k items selected from a group of n possible elements **without replacement**, such that **order does not matter**.

- **Example:** There are 24 ice cream flavors. How many ways can you pick two flavors?

base permutation logic →

$$\frac{24 \cdot 23}{2}$$

adjust for dups →

"n choose k"

$$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

From permutations to combinations

- There is a close connection between:
  - the number of permutations of k elements selected from a group of n, and
  - the number of combinations of k elements selected from a group of n

$$\text{\# combinations} = \frac{\text{\# permutations}}{\text{\# orderings of } k \text{ items}}$$

- Since # permutations = $\frac{n!}{(n-k)!}$ and # orderings of k items = $k!$, we have

$$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Combinations

In general, the number of ways to select k elements from a group of n elements such that **repetition is not allowed** and **order does not matter** is

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

The symbol $\binom{n}{k}$ is pronounced "n choose k", and is also known as the **binomial coefficient**.

Example: committees

- How many ways are there to select a president, vice president, and secretary from a group of 8 people?

  no replacement
  order matters

  $P(8,3) = 8 \cdot 7 \cdot 6$

- How many ways are there to select a committee of 3 people from a group of 8 people?

  no replacement
  order not matter

  $C(8,3) = \frac{8 \cdot 7 \cdot 6}{3!}$

- If you're ever confused about the difference between permutations and combinations, **come back to this example**.

Summary

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- A **sequence** is obtained by selecting k elements from a group of n possible elements with replacement, such that order matters.
  - Number of sequences: n^k.
- A **permutation** is obtained by selecting k elements from a group of n possible elements without replacement, such that order matters.
  - Number of permutations: $P(n,k) = \frac{n!}{(n-k)!}$.
- A **combination** is obtained by selecting k elements from a group of n possible elements without replacement, such that order does not matter.
  - Number of combinations: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.