# Module 18 - Probabability and Combinatorics Examples

DSC 40A, Summer 2023

# Agenda

- Review of combinatorics.
- Lots of examples.

# Review of combinatorics

# Combinatorics as a tool for probability

- If $S$ is a sample space consisting of equally-likely outcomes, and $A$ is an event, then $P(A) = \frac{|A|}{|S|}$.
- In many examples, this will boil down to using permutations and/or combinations to count $|A|$ and $|S|$.
- **Tip:** Before starting a probability problem, always think about what the sample space $S$ is!

# Sequences

- A **sequence** of length $k$ is obtained by selecting $k$ elements from a group of $n$ possible elements **with replacement**, such that **order matters**.

- **Example:** You roll a die 10 times. How many different sequences of results are possible?

$6^{10}$

$\underbrace{\text{1st roll} \ldots \text{2nd rol} \ldots\ldots \text{10th roll}}$

# Sequences

In general, the number of ways to select $k$ elements from a group of $n$ possible elements such that **repetition is allowed** and **order matters** is

aka "with replacement"

$$n^k.$$

# Permutations

- A **permutation** is obtained by selecting $k$ elements from a group of $n$ possible elements **without replacement**, such that **order matters**.

"repitition is not allowed"

- **Example:** How many ways are there to select a president, vice president, and secretary from a group of 8 people?

$8 \cdot 7 \cdot 6$

$$\frac{8!}{5!} \qquad \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}$$

# Permutations

- In general, the number of ways to select $k$ elements from a group of $n$ possible elements such that **repetition is not allowed** and **order matters** is

$$
\begin{aligned}
P(n,k) &= (n)(n-1)\ldots(n-k+1) \\
&= \frac{n!}{(n-k)!}
\end{aligned}
$$

# Combinations

- A **combination** is a set of $k$ items selected from a group of $n$ possible elements **without replacement**, such that **order does not matter**.

- **Example:** How many ways are there to select a committee of 3 people from a group of 8 people?

$$\frac{8!}{5!3!}$$

# Combinations

In general, the number of ways to select $k$ elements from a group of $n$ elements such that **repetition is not allowed** and **order does not matter** is

"n choose k"

$$
\begin{aligned}
C(n,k) &= \binom{n}{k} \\
&= \frac{P(n,k)}{k!} \\
&= \frac{n!}{(n-k)!k!}
\end{aligned}
$$

The symbol $\binom{n}{k}$ is pronounced "$n$ choose $k$", and is also known as the **binomial coefficient**.

## Lots of examples

| with replacement? \ order matters? | yes | no |
| --- | --- | --- |
| yes | sequences | |
| no | permutation | combination |

## Discussion Question

A domino consists of two faces, each with anywhere between 0 and 6 dots. A set of dominoes consists of every possible combination of dots on each face.

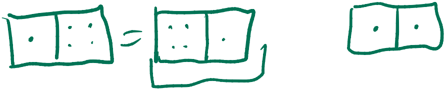
How many dominoes are in the set of dominoes?

a) $\binom{7}{2}$

b) $\binom{7}{1} + \binom{7}{2}$ why?

c) $P(7,2)$

d) $\frac{P(7,2)}{P(7,1)} 7!$



# Selecting students — overview

We're going answer the same question using several different techniques.

**Question 1:** There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

# Selecting students (Method 1: using permutations)

**Question 1:** There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

S = all possible ordered arrangements of students

label students A, B, C ... T

ex) G A T B C

$$\frac{\text{\# permutations including Avi}}{\text{\# of permutations}}$$

$$\underline{\text{denominator} \quad \frac{20!}{15!}} \quad \text{via} \quad P(20,5)$$

numerator: how many permutations include A?

if A is first...

↳ $\underset{1}{\underline{A}}\ \underset{19}{\underline{\ }}\ \underset{18}{\underline{\ }}\ \underset{17}{\underline{\ }}\ \underset{16}{\underline{\ }} \longrightarrow \underbrace{19 \cdot 18 \cdot 17 \cdot 16}_{\text{counts perms where A is 1st}}$

if A is second...

↳ $\underset{19}{\underline{\ }}\ \underset{1}{\underline{A}}\ \underset{18}{\underline{\ }}\ \underset{17}{\underline{\ }}\ \underset{16}{\underline{\ }} \longrightarrow \underbrace{19 \cdot 18 \cdot 17 \cdot 16}_{\text{counts perms where A is 2nd}}$

↳ and so on to A being 5th

thus... $5 \cdot 19 \cdot 18 \cdot 17 \cdot 16 = 5 \cdot P(19,4) = 5 \cdot \frac{19!}{15!}$

$$\frac{\text{num}}{\text{denom}} = \frac{5 \cdot \frac{19!}{15!}}{\frac{20!}{15!}} = 5 \cdot \frac{19!}{\cancel{15!}} \cdot \frac{\cancel{15!}}{20!}$$

$$= 5 \cdot \frac{1}{20}$$

$$= \frac{1}{4}$$

## Selecting students (Method 2: using permutations and the complement)

**Question 1:** There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

S = perms as before

last time...

$$\frac{\#\text{ perms incl } A}{\#\text{ all permutations}} = \frac{\overset{P(20,5)}{\#\text{ all permutations}} - \text{permutations without } A\,?}{P(20,5)\ \#\text{ all permutations}}$$

exclude Avi from sample n

$P(19,5)$

$$\frac{P(20,5) - P(19,5)}{P(20,5)}$$

# Selecting students (Method 3: using combinations)

**Question 1:** There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

$S =$ all sets of 5 selected students

remaining slots once Avi is selected

$$\frac{\#\text{ of sets including Avi}}{\#\text{ of sets}} \quad \begin{matrix} \leftarrow \binom{19}{4} \\ \leftarrow \binom{20}{5} = \frac{20!}{15!\,5!} \end{matrix}$$

# Selecting students (Method 3: using combinations)

**Question 1, Part 1 (Denominator):** If you draw a sample of size 5 at random without replacement from a population of size 20, how many different **sets** of individuals could you draw?

$$C(20,5) = \binom{20}{5} = \frac{20!}{15!\,5!}$$

# Selecting students (Method 3: using combinations)

**Question 1, Part 2 (Numerator):** If you draw a sample of size 5 at random without replacement from a population of size 20, how many different **sets** of individuals include Avi?

$$C(19,4) = \binom{19}{4} = \frac{19!}{15!\,4!}$$

# Selecting students (Method 3: using combinations)

**Question 1:** There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

$$\frac{\frac{19!}{15!\,4!}}{\frac{20!}{15!\,5!}} = \frac{\cancel{19!}}{\cancel{15!}\,\cancel{4!}} \cdot \frac{\cancel{15!}\,\cancel{5!}^{\,5}}{\cancel{20!}_{\,20}}$$

$$= \frac{5}{20}$$

$$= \frac{1}{4}$$

# Selecting students (Method 4: “the easy way”)

**Question 1:** There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

$\underbrace{\_\ \_\ \_\ \_\ \_}_{\text{selected}}\ \underbrace{\_\ \_\ \_\ \_\ \_}\ \underbrace{\_\ \_\ \_\ \_\ \_}\ \underbrace{\_\ \_\ \_\ \_\ \_}$

A

A

A

A

# Summary

## Summary

use these to define cardinality

- A **sequence** is obtained by selecting $k$ elements from a group of $n$ possible elements with replacement, such that order matters.
  - Number of sequences: $n^k$.

- A **permutation** is obtained by selecting $k$ elements from a group of $n$ possible elements without replacement, such that order matters.
  - Number of permutations: $P(n,k) = \frac{n!}{(n-k)!}$.

- A **combination** is obtained by selecting $k$ elements from a group of $n$ possible elements without replacement, such that order does not matter.
  - Number of combinations: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.