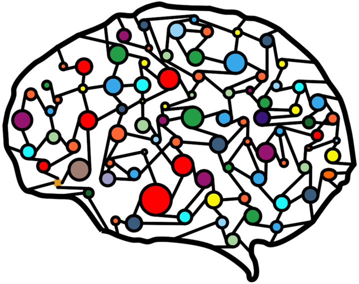
Module 19 - More Probabability and Combinatorics Examples



DSC 40A, Summer 2023

Agenda

- Lots of examples.

Last time

Last time we answered the same question using several different techniques.

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

1/4

With vs. without replacement

Discussion Question

We've determined that a probability that a random sample of 5 students from a class of 20 **without replacement** contains Avi (one student in particular) is $\frac{1}{4}$.

Suppose we instead sampled **with replacement**. Would the resulting probability be equal to, greater than, or less than $\frac{1}{4}$?

a) Equal to
b) Greater than
c) Less than

without replacement

$P(\text{Avi on first pick}) = 1/20$

$P(\text{Avi on second} \mid \text{Avi not first}) = 1/19$

and so on...

with replacement

$P(\text{Avi first}) = 1/20$

$P(\text{Avi second} \mid \text{Avi not first}) = \frac{1}{20}$

Extreme case: we will select 20 from 20

without replacement: $P(\text{Avi}) = 1$ with replacement: $P(\text{Avi}) < 1$

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odds of not selecting avi

$\underline{19/20}\ \underline{19/20}\ \underline{19/20}\ \underline{19/20}\ \underline{19/20}$

$$1 - \left(\frac{19}{20}\right)^5 = 0.23$$

Art supplies

Question 2, Part 1: We have 12 art supplies: 5 markers and 7 crayons. In how many ways can we select 4 art supplies?

order does not matter

no replacement

$C(12,4)$

Art supplies

Question 2, Part 2: We have 12 art supplies: 5 markers and 7 crayons. In how many ways can we select 4 art supplies such that we have...

1. 2 markers and 2 crayons?
2. 3 markers and 1 crayon?

$C(5,2) * C(7,2)$

$C(5,3) * (7,1)$

Art supplies

Question 2, Part 3: We have 12 art supplies: 5 markers and 7 crayons. We randomly select 4 art supplies. What's the probability that we selected at least 2 markers?

S = all sets of 4 supplies

$|S| = C(12,4)$

0 markers $\binom{5}{0}\binom{7}{4}$

1 marker $\binom{5}{1}\binom{7}{3}$

2 markers $\binom{5}{2}\binom{7}{2}$

3 markers $\binom{5}{3}\binom{7}{1}$

4 markers $\binom{5}{4}\binom{7}{0}$

$$P(\text{at least 2 markers}) = \frac{\#\text{ of sets of 4 supplies w/ min 2 markers}}{C(12,4)}$$

$$\frac{\binom{5}{2}\binom{7}{2} + \binom{5}{3}\binom{7}{1} + \binom{5}{4}\binom{7}{0}}{\binom{12}{4}}$$

Fair coin

Question 3: Suppose we flip a **fair coin** 10 times.

1. What is the probability that we see the specific sequence THTTHTHHTH? ← ordered with replacement
2. What is the probability that we see an equal number of heads and tails?

$\left(\frac{1}{2}\right)^{10}$

order matters? no

choose idx for tails

sequences with 5 H, 5 T → $\frac{C(10,5)}{2^{10}}$

Unfair coin

Question 4: Suppose we flip an **unfair coin** 10 times. The coin is biased such that for each flip, $P(\text{heads}) = \frac{1}{3}$.

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

T H T T H T H H T H $\left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^5$

2/3 1/3 2/3 2/3 1/3 2/3 1/3 1/3 2/3 1/3

S = all sequences of 10 rolls

$C(10,5) * \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^5$

Deck of cards

- There are 52 cards in a standard deck (4 suits, 13 values).

4 suits 13 values

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- In poker, each player is dealt 5 cards, called a **hand**. The order of cards in a hand does not matter.

Deck of cards

1. How many 5 card hands are there in poker?

   $C(52, 5)$

2. How many 5 card hands are there where all cards are of the same suit (a **flush**)?

   ex) 5♡, 6♡, K♡, A♡, J♡

   what suit? 4 choices

   $C(13,5)$ combinations of values

   $4 \cdot C(13,5)$

3. How many 5 card hands are there that include a **four-of-a-kind** (four cards of the same value)?

ex) $\underbrace{A\diamondsuit,\ A\heartsuit,\ A\clubsuit,\ A\spadesuit}_{\text{cardinality?}},\ 2\diamondsuit$

$13 \cdot 48$

$(12 \cdot 4)$

4. How many 5 card hands are there that have a **straight** (all card values consecutive)?

in poker, A counts as high or low

A2345, 23456, 34567, 45678, 56789, 678910, 78910J, 8910JQ, 910JQK, 10JQKA

1, 2, 3, 4, 5, 6, 7, 8, 9, 10

4^5

$10 \cdot 4^5$

5. How many 5 card hands are there that are a **straight flush** (all card values consecutive and of the same suit)?

same as before

$$10 \cdot 4 = 40$$

now choose a suit

6. How many 5 card hands are there that include exactly **one pair** (values aabcd)?

13 possible values for pair

C(4,2) possible suits for pair
⇒ 6

C(12,3) values for the remaining cards

4^3 suits for remaining cards
⇒ 64

$13 \cdot 6 \cdot 220 \cdot 64 = 1098240$

Summary

Summary

- A **sequence** is obtained by selecting k elements from a group of n possible elements with replacement, such that order matters.
  - Number of sequences: n^k.

- A **permutation** is obtained by selecting k elements from a group of n possible elements without replacement, such that order matters.
  - Number of permutations: $P(n,k) = \frac{n!}{(n-k)!}$.

- A **combination** is obtained by selecting k elements from a group of n possible elements without replacement, such that order does not matter.
  - Number of combinations: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.