Module 20 – Law of Total Probability and Bayes' Theorem

DSC 40A, Summer 2023

Agenda

- Partitions and the Law of Total Probability.
- Bayes' Theorem.

Law of Total Probability

Example: getting to school

You conduct a survey where you ask students two questions.

1. How did you get to campus today? Trolley, bike, or drive? (Assume these are the only options.)
2. Were you late?

	Late	Not Late
Trolley	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

	Late	Not Late	
Trolley	0.06	0.24	0.30
Bike	0.03	0.07	0.10
Drive	0.36	0.24	0.60
	0.45	0.55	

Discussion Question

What's the probability that a randomly selected person was late?

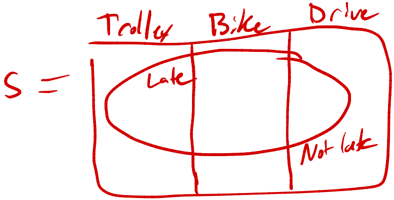
a) 0.24
b) 0.30
c) 0.45
d) 0.50
e) None of the above

Example: getting to school

	Late	Not Late
Trolley	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

- Since everyone either takes the trolley, bikes, or drives to school, we have

$$P(\text{Late}) = P(\text{Late} \cap \text{Trolley}) + P(\text{Late} \cap \text{Bike}) + P(\text{Late} \cap \text{Drive})$$



	Late	Not Late
Trolley	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

$= 0.30$

$= \frac{0.06}{0.30}$

$=$

Discussion Question

Avi took the trolley to school. What is the probability that he was late?

a) 0.06
b) 0.2
c) 0.25
d) 0.45
e) None of the above

$P(\text{late} \mid \text{trolley}) = 0.2$

$= \frac{P(\text{late} \cap \text{trolley})}{P(\text{trolley})}$

$P(\text{trolley} \cap \text{late}) + P(\text{trolley} \cap \text{not late})$

Example: getting to school

	Late	Not Late
Trolley	0.06	0.24
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Drive	0.36	0.24

- Since everyone either takes the trolley, bikes, or drives to school, we have

$$P(\text{Late}) = P(\text{Late} \cap \text{Trolley}) + P(\text{Late} \cap \text{Bike}) + P(\text{Late} \cap \text{Drive})$$

- Another way of expressing the same thing:

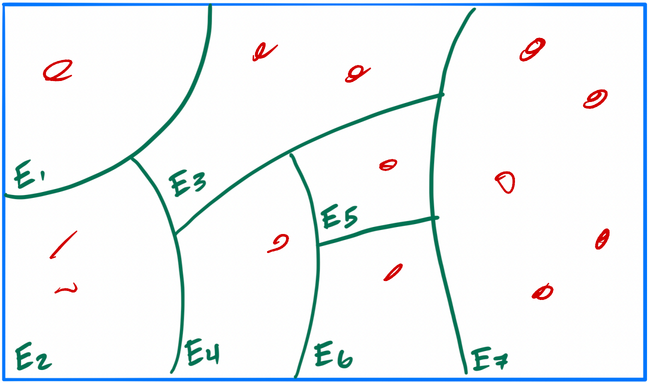
$$P(\text{Late}) = P(\text{Trolley})\, P(\text{Late}|\text{Trolley}) + P(\text{Bike})\, P(\text{Late}|\text{Bike}) + P(\text{Drive})\, P(\text{Late}|\text{Drive})$$

Partitions

- A set of events $E_1, E_2, ..., E_k$ is a **partition** of S if
  - $P(E_i \cap E_j) = 0$ for all pairs $i \neq j$.

    any pair of events
  - $P(E_1 \cup E_2 \cup ... \cup E_k) = 1$.
    - Equivalently, $P(E_1) + P(E_2) + ... + P(E_k) = 1$.
- In other words, $E_1, E_2, ..., E_k$ is a partition of S if every outcome s in S is in **exactly** one event E_i.

Partitions, visualized



Example partitions

- In getting to school, the events Trolley, Bike, and Drive.
- In getting to school, the events Late and Not Late.
- In selecting an undergraduate student at random, the events Freshman, Sophomore, Junior, and Senior.
- In rolling a die, the events Even and Odd.
- In drawing a card from a standard deck of cards, the events Spades, Clubs, Hearts, and Diamonds.

Example partitions

- In getting to school, the events Trolley, Bike, and Drive.
- In getting to school, the events Late and Not Late.
- In selecting an undergraduate student at random, the events Freshman, Sophomore, Junior, and Senior.
- In rolling a die, the events Even and Odd.
- In drawing a card from a standard deck of cards, the events Spades, Clubs, Hearts, and Diamonds.
- **Special case**: any event A and its complement $\bar{A}$.

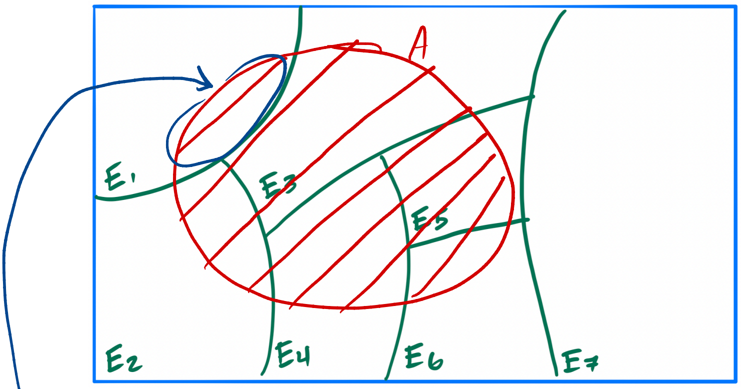
The Law of Total Probability

- If A is an event and $E_1, E_2, \ldots, E_k$ is a **partition** of S, then

$$P(A) = P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_k)$$

$$= \sum_{i=1}^{k} P(A \cap E_i)$$

The Law of Total Probability, visualized



$$P(A) = P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_6) + P(A \cap E_7)$$

↳ would eval to 0

The Law of Total Probability

- If A is an event and $E_1, E_2, \ldots, E_k$ is a **partition** of S, then

$$P(A) = P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_k)$$
$$= \sum_{i=1}^{k} P(A \cap E_i)$$

- Since $P(A \cap E_i) = P(E_i) \cdot P(A|E_i)$ by the multiplication rule, an equivalent formulation is

$$P(A) = P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2) + \ldots + P(E_k) \cdot P(A|E_k)$$
$$= \sum_{i=1}^{k} P(E_i) \cdot P(A|E_i)$$

	Late	Not Late
Trolley	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

0.45

Discussion Question

Lauren is late to school. What is the probability that she took the trolley? Choose the best answer.

a) Close to 0.05
b) Close to 0.15
c) Close to 0.3
d) Close to 0.4

$$P(\text{trolley} \mid \text{late}) = \frac{P(\text{Trolley} \cap \text{Late})}{P(\text{Late})} = \frac{0.06}{0.45}$$

Bayes' Theorem

Example: getting to school

- Now suppose you don't have that entire table. Instead, all you know is
  - $P(\text{Late}) = 0.45$.
  - $P(\text{Trolley}) = 0.3$.
  - $P(\text{Late}|\text{Trolley}) = 0.2$.

Flipped

- Can you still find $P(\text{Trolley}|\text{Late})$?

$$P(\text{trolley}|\text{late}) = \frac{P(\text{Trolley} \cap \text{Late})}{P(\text{Late})}$$

$$= \frac{P(\text{trolley}) \cdot P(\text{late}|\text{trolley})}{P(\text{late})}$$

$$= \frac{0.3 \cdot 0.2}{0.45} = \frac{0.06}{0.45} = 0.13\overline{3}$$

Bayes' Theorem

- Recall that the multiplication rule states that

$$P(A \cap B) = P(A) \cdot P(B|A)$$

- It also states that

$$P(B \cap A) = P(B) \cdot P(A|B)$$

- But since $A \cap B = B \cap A$, we have that

$$P(A) \cdot P(B|A) = P(B) \cdot P(A|B)$$

- Re-arranging yields **Bayes' Theorem**:

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}$$

Bayes' Theorem and the Law of Total Probability

- Bayes' Theorem:

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}$$

Bayes

- Recall from earlier, for any sample space S, B and $\bar{B}$ partition S. Using the Law of Total Probability, we can re-write $P(A)$ as

$$P(A) = P(A \cap B) + P(A \cap \bar{B}) = P(B) \cdot P(A|B) + P(\bar{B}) \cdot P(A|\bar{B})$$

Bayes' Theorem and the Law of Total Probability

- Bayes' Theorem:

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}$$

- Recall from earlier, for any sample space S, B and $\bar{B}$ partition S. Using the Law of Total Probability, we can re-write $P(A)$ as

$$P(A) = P(A \cap B) + P(A \cap \bar{B}) = P(B) \cdot P(A|B) + P(\bar{B}) \cdot P(A|\bar{B})$$

- This means that we can re-write Bayes' Theorem as

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(B) \cdot P(A|B) + P(\bar{B}) \cdot P(A|\bar{B})}$$

Example: drug test

A manufacturer claims that its drug test will **detect steroid use 95% of the time**. What the company does not tell you is that 15% of all steroid-free individuals also test positive (the false positive rate). Suppose 10% of the Tour de France bike racers use steroids and your favorite cyclist just tested positive. What's the probability that they used steroids?

$P(A|B)$ (95%), $P(A|\bar{B})$ (15%), $P(B)$ (10%), $P(\bar{B}) = .9$

$P(\text{use steroids} \mid \text{positive test})$, with B = use steroids, A = positive test

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}$$

$$= \frac{P(B) \cdot P(A|B)}{P(A \cap B) + P(A \cap \bar{B})}$$

$$= \frac{P(B) \cdot P(A|B)}{P(B) \cdot P(A|B) + P(\bar{B}) \cdot P(A|\bar{B})}$$

$$\frac{.1 \cdot .95}{.1 \cdot .95 + .9 \cdot .15} \approx 0.41$$

Example: taste test

I = In-n-out
S = shake shack
F = five guys
C = correct

- Your friend claims to be able to correctly guess what restaurant a burger came from, after just one bite.
- The probability that she correctly identifies an In-n-Out Burger is 0.55, a Shake Shack burger is 0.75, and a Five Guys burger is 0.6.
- You buy 5 In-n-Out burgers, 4 Shake Shack burgers, and 1 Five Guys burger, choose one of the burgers randomly, and give it to her.
- **Question:** Given that she guessed it correctly, what's the probability she ate a Shake Shack burger?

$P(C|I) = .55$ $P(I) = .5$
$P(C|S) = .75$ $P(S) = .4$
$P(C|F) = .6$ $P(F) = .1$

$$P(S|C) = \frac{P(S)\cdot P(C|S)}{P(C)}$$

$$= \frac{P(S)\cdot P(C|S)}{P(I)\cdot P(C|I) + P(S)\cdot P(C|S) + P(F)\cdot P(C|F)}$$

$$\frac{.4\cdot .75}{.5\cdot .55 + .4\cdot .75 + .1\cdot .6} \approx 0.47$$

Summary

Summary

- A set of events $E_1, E_2, \ldots, E_k$ is a **partition** of S if each outcome in S is in exactly one E_i.
- The Law of Total Probability states that if A is an event and $E_1, E_2, \ldots, E_k$ is a **partition** of S, then

$$\begin{aligned} P(A) &= P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2) + \ldots + P(E_k) \cdot P(A|E_k) \\ &= \sum_{i=1}^{k} P(E_i) \cdot P(A|E_i) \end{aligned}$$

- Bayes' Theorem states that

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}$$

- We often re-write the denominator $P(A)$ in Bayes' Theorem using the Law of Total Probability.