Module 21 – Independence

DSC 40A, Summer 2023

Announcements

- Homework 3 is due **tomorrow at 11:59pm**.
- Great source of practice problems for recent content: stat88.org/textbook.
- Also check out the Probability Roadmap on the resources tab of the course website.
- Reminder: Final will be closed books/notes/electronics/web. You will be allowed to keep with you two A4-sized sheets (four sides) with any content you want.
- Extra Credit: If 90% or greater response rate on SET evaluations (formerly CAPE) by August 4, class gets a 0.5% boost to overall grade.

Agenda

- Recap of Module 20.
- Independence.

Last time

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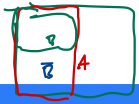
- A set of events $E_1, E_2, ..., E_k$ is a **partition** of S if each outcome in S is in exactly one E_i.
- The **Law of Total Probability** states that if A is an event and $E_1, E_2, ..., E_k$ is a partition of S, then

$$P(A) = P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2) + ... + P(E_k) \cdot P(A|E_k)$$
$$= \sum_{i=1}^{k} P(E_i) \cdot P(A|E_i)$$

- **Bayes' Theorem** states that

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}$$

- We often re-write the denominator $P(A)$ in Bayes' Theorem using the Law of Total Probability.



Discussion Question

Consider any two events A and B. Choose the expression that's equivalent to

$$P(B|A) + P(\bar{B}|A).$$

$$\frac{P(A \cap B)}{P(A)} + \frac{P(A \cap \bar{B})}{P(A)}$$

a) $P(A)$
b) $1 - P(B)$
c) $P(B)$
d) $P(\bar{B})$
e) 1

Example: prosecutor's fallacy

A bank was robbed yesterday by one person. Consider the following facts about the crime:

- The person who robbed the bank wore Nikes.
- Of the 10,000 other people who came to the bank yesterday, only 10 of them wore Nikes.

The prosecutor finds the prime suspect, and states that "given this evidence, the chance that the prime suspect was not at the crime scene is 1 in 1,000". ← prob of innocent – derived from $\frac{10}{10000}$
$P(\text{nikes} \mid \text{innocent})$

1. What is wrong with this statement? $\neq P(\text{innocent} \mid \text{nikes})$

2. Find the probability that the prime suspect is guilty given only the evidence in the exercise.

	guilty	innocent	
nikes	1	10	11 total
no nikes	0	9,990	9,990 total
			10,001 total

$$P(\text{innocent} \mid \text{nikes}) = \frac{10}{11}$$

$$\frac{P(\text{nikes} \wedge \text{innocent})}{P(\text{nikes})} \qquad \frac{10/\cancel{10001}}{11/\cancel{10001}}$$

Independence

Updating probabilities

B = innocent
A = nikes
prob innocent before we knew nikes
10,000/10,001

- Bayes' theorem describes how to update the probability of one event, given that another event has occurred.

old/prior

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}$$

← multiply by this to update probability

  - $P(B)$ can be thought of as the "prior" probability of B occurring, before knowing anything about A.
  - $P(B|A)$ is sometimes called the "posterior" probability of B occurring, given that A occurred.
- What if knowing that A occurred doesn't change the probability that B occurs? In other words, what if

$$\underline{P(B|A) = P(B)}$$

Independent events

- A and B are **independent events** if one event occurring does not affect the chance of the other event occurring.

$$P(B|A) = P(B) \qquad\qquad P(A|B) = P(A)$$

- Otherwise, A and B are **dependent events**.

- Using Bayes' theorem, we can show that if one of the above statements is true, then so is the other.

Independent events

- **Equivalent definition:** A and B are independent events if

$$P(A \cap B) = P(A) \cdot P(B)$$

- To check if A and B are independent, use whichever is easiest:
  - $P(B|A) = P(B)$.
  - $P(A|B) = P(A)$.
  - $P(A \cap B) = P(A) \cdot P(B)$.

$P(A \cap B) = P(A) \cdot P(B|A)$

Mutual exclusivity and independence

events don't overlap,
one precludes the other

$P(A|B) = P(A)$
$P(B|A) = P(B)$
$P(A \cap B) = P(A) \cdot P(B)$

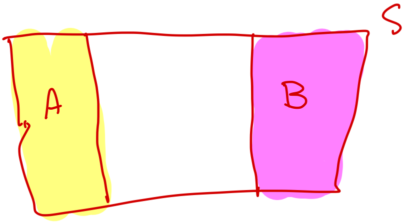
Discussion Question

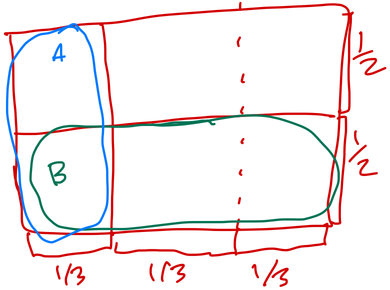
Suppose A and B are two events with non-zero probabilities. Is it possible for A and B to be both mutually exclusive and independent?

a) Yes
b) No

If mutually exclusive, $P(B|A) = 0$

Visualize mutual exclusivity





$P(B|A) = 1/2$

$P(B) = 1/2$

$P(A|B) = 1/3$

$P(A) = 1/3$

Example: Venn diagrams

addition rule

$P(A \cup C) = P(A) + P(C) - P(A \cap C)$

A and C are independent, so...

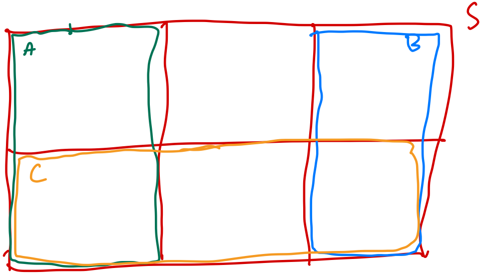
$P(A) \cdot P(C)$

$= P(A) + P(C) - P(A) \cdot P(C)$

For three events A, B, and C, we know that

- A and C are independent,
- B and C are independent,
- A and B are mutually exclusive,
- $P(A \cup C) = \frac{2}{3}$, $P(B \cup C) = \frac{3}{4}$, $P(A \cup B \cup C) = \frac{11}{12}$.

Find $P(A)$, $P(B)$, and $P(C)$.



(not to scale...)

mutually exclusive
so evaluates to 0

$$P(A \cup C) = 2/3 = P(A) + P(C) - P(A)\cdot P(C)$$

$$P(B \cup C) = 3/4 = P(B) + P(C) - P(B)\cdot P(C)$$

$$P(A \cup B \cup C) = 11/12 = P(A) + P(B) + P(C) - P(A)\cdot P(C) - P(B)\cdot P(C) \underbrace{- \cancel{P(A)\cdot P(B)}}_{0}$$

$$\cancel{P(A)} + \cancel{P(B)} + P(C) - \left(\cancel{P(A)} + P(C) - 2/3\right) - \left(\cancel{P(B)} + P(C) - 3/4\right) = \frac{11}{12}$$

$$-P(C) + \frac{2}{3} + \frac{3}{4} = \frac{11}{12}$$

$$+P(C) = -\frac{11}{12} + \frac{2}{3} + \frac{3}{4}$$

$$P(C) = \frac{8}{12} + \frac{9}{12} - \frac{11}{12}$$

$$= \frac{6}{12} = \frac{1}{2}$$

Summary

Summary

- Two events A and B are **independent** when knowledge of one event does not change the probability of the other event.
- There are there equivalent definitions of independence:
  - $P(B|A) = P(B)$
  - $P(A|B) = P(A)$
  - $P(A \cap B) = P(A) \cdot P(B)$