Module 23 – Naive Bayes

DSC 40A, Summer 2023

Agenda

- Classification.
- Classification and conditional independence.
- Naive Bayes.

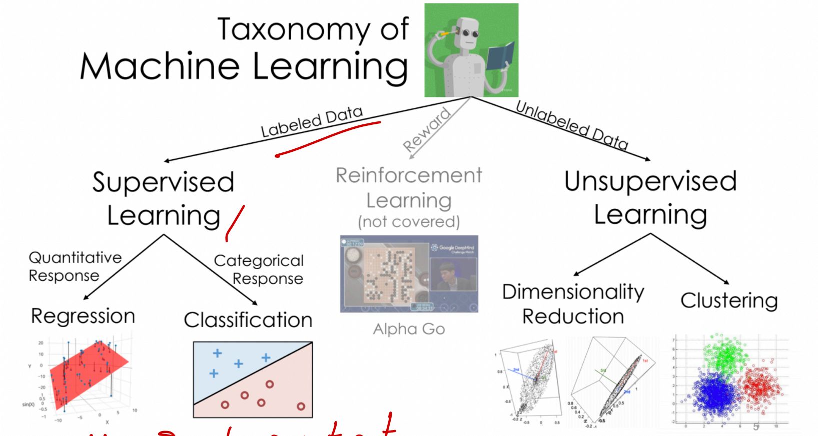
↳ probabilistic approach to supervised learning

Recap: Bayes' theorem, independence, and conditional independence

- Bayes' theorem: $P(A|B) = \frac{P(A)P(B|A)}{P(B)}$.
- A and B are **independent** if $P(A \cap B) = P(A) \cdot P(B)$.
- A and B are **conditionally independent** given C if $P((A \cap B)|C) = P(A|C) \cdot P(B|C)$.
  - In general, there is no relationship between independence and conditional independence.

Classification

Taxonomy of machine learning



binary → cat vs not cat

multiclass → cat vs dog vs other

Classification problems

- Like with regression, we're interested in making predictions based on data (called **training data**) for which we know the value of the response variable.

- The difference is that the response variable is now **categorical**.

- Categories are called **classes**.

- Example classification problems:
  - class imbalance → Deciding whether a patient has kidney disease. → binary
  - multiclass, balanced → Identifying handwritten digits.
  - Determining whether an avocado is ripe. → binary
  - class imbalance → Predicting whether credit card activity is fraudulent.

Example: avocados

out of sample

You have a green-black avocado, and want to know if it is ripe.

color	ripeness	
bright green	unripe	
green-black	ripe	R
purple-black	ripe	
green-black	unripe	U
purple-black	ripe	
bright green	unripe	
green-black	ripe	R
purple-black	ripe	
green-black	ripe	R
green-black	unripe	U
purple-black	ripe	

training data

Question: Based on this data, would you predict that your avocado is ripe or unripe?

5 green black

3/5 ripe

2/5 unripe

Example: avocados

You have a green-black avocado, and want to know if it is ripe. Based on this data, would you predict that your avocado is ripe or unripe?

color	ripeness
bright green	unripe
green-black	ripe
purple-black	ripe
green-black	unripe
purple-black	ripe
bright green	unripe
green-black	ripe
purple-black	ripe
green-black	ripe
green-black	unripe
purple-black	ripe

Strategy: Calculate two probabilities:

$$P(\text{ripe}|\text{green-black}) = \frac{3}{5}$$

$$P(\text{unripe}|\text{green-black}) = \frac{2}{5}$$

Then, predict the class with a **larger** probability.

Estimating probabilities

- We would like to determine $P(\text{ripe}|\text{green-black})$ and $P(\text{unripe}|\text{green-black})$ for all avocados in the universe.

- All we have is a single dataset, which is a **sample** of all avocados in the universe.

- We can estimate these probabilities by using sample proportions.

$$P(\text{ripe}|\text{green-black}) \approx \frac{\#\text{ ripe green-black avocados in sample}}{\#\text{ green-black avocados in sample}}$$

- Per the **law of large numbers** in DSC 10, larger samples lead to more reliable estimates of population parameters.

Example: avocados

You have a green-black avocado, and want to know if it is ripe. Based on this data, would you predict that your avocado is ripe or unripe?

color	**ripeness**
bright green	unripe
green-black	ripe
purple-black	ripe
green-black	unripe
purple-black	ripe
bright green	unripe
green-black	ripe
purple-black	ripe
green-black	ripe
green-black	unripe
purple-black	ripe

$P(\text{ripe}|\text{green-black}) = 3/5$

$P(\text{unripe}|\text{green-black}) = 2/5$

Bayes' theorem for classification

- Suppose that A is the event that an avocado has certain features, and B is the event that an avocado belongs to a certain class. Then, by Bayes' theorem:

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}$$

Class: unripe or ripe

features

- More generally:

ripe

green black | ripe

$$P(\text{class}|\text{features}) = \frac{P(\text{class}) \cdot P(\text{features}|\text{class})}{P(\text{features})}$$

green black

- What's the point?
  - Usually, it's not possible to estimate $P(\text{class}|\text{features})$ directly from the data we have.
  - Instead, we have to estimate $P(\text{class})$, $P(\text{features}|\text{class})$, and $P(\text{features})$ separately.

Example: avocados

You have a green-black avocado, and want to know if it is ripe. Based on this data, would you predict that your avocado is ripe or unripe?

color	ripeness
bright green	unripe
green-black	ripe
purple-black	ripe
green-black	unripe
purple-black	ripe
bright green	unripe
green-black	ripe
purple-black	ripe
green-black	ripe
green-black	unripe
purple-black	ripe

$$P(\text{class}|\text{features}) = \frac{P(\text{class})\cdot P(\text{features}|\text{class})}{P(\text{features})}$$

$$P(\text{ripe} \mid \text{green-black}) = \frac{P(\text{ripe}) \cdot P(\text{greenblack} \mid \text{ripe})}{P(\text{greenblack})} = \frac{7/11 \cdot 3/7}{5/11}$$

$$P(\text{unripe} \mid \text{green-black}) \quad \frac{4/11 \cdot 2/4}{5/11} = \frac{2}{5}$$

Example: avocados

You have a green-black avocado, and want to know if it is ripe. Based on this data, would you predict that your avocado is ripe or unripe?

color	ripeness
bright green	unripe
green-black	ripe
purple-black	ripe
green-black	unripe
purple-black	ripe
bright green	unripe
green-black	ripe
purple-black	ripe
green-black	ripe
green-black	unripe
purple-black	ripe

$$P(\text{class}|\text{features}) = \frac{P(\text{class}) \cdot P(\text{features}|\text{class})}{P(\text{features})}$$

Example: avocados

You have a green-black avocado, and want to know if it is ripe. Based on this data, would you predict that your avocado is ripe or unripe?

color	**ripeness**
bright green	unripe
green-black	ripe
purple-black	ripe
green-black	unripe
purple-black	ripe
bright green	unripe
green-black	ripe
purple-black	ripe
green-black	ripe
green-black	unripe
purple-black	ripe

$$P(\text{class}|\text{features}) = \frac{P(\text{class}) \cdot P(\text{features}|\text{class})}{P(\text{features})}$$

Shortcut: Both probabilities have the same denominator. The larger one is the one with the larger numerator.

$$P(\text{ripe}|\text{green-black})$$

$$P(\text{unripe}|\text{green-black})$$

Classification and conditional independence

Example: avocados, but with more features

color	softness	variety	ripeness
bright green	firm	Zutano	unripe
green-black	medium	Hass	ripe
purple-black	firm	Hass	ripe
green-black	medium	Hass	unripe
purple-black	soft	Hass	ripe
bright green	firm	Zutano	unripe
green-black	soft	Zutano	ripe
purple-black	soft	Hass	ripe
green-black	soft	Zutano	ripe
green-black	firm	Hass	unripe
purple-black	medium	Hass	ripe

softness color variety

You have a firm green-black Zutano avocado. Based on this data, would you predict that your avocado is ripe or unripe?

Example: avocados, but with more features

color	softness	variety	ripeness
bright green	firm	Zutano	unripe
green-black	medium	Hass	ripe
purple-black	firm	Hass	ripe
green-black	medium	Hass	unripe
purple-black	soft	Hass	ripe
bright green	firm	Zutano	unripe
green-black	soft	Zutano	ripe
purple-black	soft	Hass	ripe
green-black	soft	Zutano	ripe
green-black	firm	Hass	unripe
purple-black	medium	Hass	ripe

You have a firm green-black Zutano avocado. Based on this data, would you predict that your avocado is ripe or unripe?

Strategy: Calculate $P(\text{ripe}|\text{features})$ and $P(\text{unripe}|\text{features})$ and choose the class with the **larger** probability.

$$P(\text{ripe}|\text{firm, green-black, Zutano})$$

$$P(\text{unripe}|\text{firm, green-black, Zutano})$$

Example: avocados, but with more features

color	softness	variety	ripeness
bright green	firm	Zutano	unripe
green-black	medium	Hass	ripe
purple-black	firm	Hass	ripe
green-black	medium	Hass	unripe
purple-black	soft	Hass	ripe
bright green	firm	Zutano	unripe
green-black	soft	Zutano	ripe
purple-black	soft	Hass	ripe
green-black	soft	Zutano	ripe
green-black	firm	Hass	unripe
purple-black	medium	Hass	ripe

You have a firm green-black Zutano avocado. Based on this data, would you predict that your avocado is ripe or unripe?

Issue: We have not seen a firm green-black Zutano avocado before.

This means that $P(\text{ripe}|\text{firm, green-black, Zutano})$ and $P(\text{unripe}|\text{firm, green-black, Zutano})$ are undefined.

A simplifying assumption

- We want to find $P(\text{ripe}|\text{firm, green-black, Zutano})$, but there are no firm green-black Zutano avocados in our dataset.

- Bayes' theorem tells us this probability is equal to

$$P(\text{ripe}|\text{firm, green-black, Zutano}) = \frac{P(\text{ripe}) \cdot P(\text{firm, green-black, Zutano}|\text{ripe})}{P(\text{firm, green-black, Zutano})}$$

- **Key idea:** **Assume** that features are **conditionally independent** given a class (e.g. ripe).

$$P(\text{firm, green-black, Zutano}|\text{ripe}) = P(\text{firm}|\text{ripe}) \cdot P(\text{green-black}|\text{ripe}) \cdot P(\text{Zutano}|\text{ripe})$$

Example: avocados, but with more features

color	softness	variety	ripeness
bright green	firm	Zutano	unripe
green-black	medium	Hass	ripe
purple-black	firm	Hass	ripe
green-black	medium	Hass	unripe
purple-black	soft	Hass	ripe
bright green	firm	Zutano	unripe
green-black	soft	Zutano	ripe
purple-black	soft	Hass	ripe
green-black	soft	Zutano	ripe
green-black	firm	Hass	unripe
purple-black	medium	Hass	ripe

You have a firm green-black Zutano avocado. Based on this data, would you predict that your avocado is ripe or unripe?

$$P(\text{ripe}|\text{firm, green-black, Zutano}) = \frac{P(\text{ripe}) \cdot P(\text{firm, green-black, Zutano}|\text{ripe})}{P(\text{firm, green-black, Zutano})}$$

$= P(\text{ripe}) \cdot P(\text{firm}|\text{ripe}) \cdot P(\text{gb}|\text{ripe}) \cdot P(\text{zutano}|\text{ripe})$

$\frac{7}{11} \cdot \frac{1}{7} \cdot \frac{3}{7} \cdot \frac{2}{7} = \frac{42}{3773} = \frac{6}{539}$

$P(\text{unripe}) \cdot P(\text{firm}|\text{unripe}) \cdot P(\text{gb}|\text{unripe}) \cdot P(\text{zutano}|\text{unripe})$

$\frac{4}{11} \quad \frac{3}{4} \cdot \frac{2}{4} \cdot \frac{2}{4}$

$\frac{3}{44}$

Example: avocados, but with more features

color	softness	variety	ripeness
bright green	firm	Zutano	unripe
green-black	medium	Hass	ripe
purple-black	firm	Hass	ripe
green-black	medium	Hass	unripe
purple-black	soft	Hass	ripe
bright green	firm	Zutano	unripe
green-black	soft	Zutano	ripe
purple-black	soft	Hass	ripe
green-black	soft	Zutano	ripe
green-black	firm	Hass	unripe
purple-black	medium	Hass	ripe

You have a firm green-black Zutano avocado. Based on this data, would you predict that your avocado is ripe or unripe?

$$P(\text{unripe}|\text{firm, green-black, Zutano}) = \frac{P(\text{unripe}) \cdot P(\text{firm, green-black, Zutano}|\text{unripe})}{P(\text{firm, green-black, Zutano})}$$

Conclusion

- The numerator of $P(\text{ripe}|\text{firm, green-black, Zutano})$ is $\frac{6}{539}$.
- The numerator of $P(\text{unripe}|\text{firm, green-black, Zutano})$ is $\cancel{\frac{6}{88}}$. 3/44
  - Both probabilities have the same denominator, $P(\text{firm, green-black, Zutano})$.
  - Since we're just interested in seeing which one is larger, we can ignore the denominator and compare numerators.
- Since the numerator for unripe is **larger** than the numerator for ripe, we **predict that our avocado is unripe**.

Naive Bayes

Naive Bayes classifier

- We want to predict a class, given certain features.
- Using Bayes' theorem, we write

$$P(\text{class}|\text{features}) = \frac{P(\text{class}) \cdot P(\text{features}|\text{class})}{P(\text{features})}$$

- For each class, we compute the numerator using the **naive assumption of conditional independence of features given the class**.
- We estimate each term in the numerator based on the training data.
- We predict the class with the largest numerator.
  - Works if we have multiple classes, too!

Dictionary

Definitions from Oxford Languages · Learn more

na·ive

adjective

(of a person or action) showing a lack of experience, wisdom, or judgment.
"the rather naive young man had been totally misled"

- (of a person) natural and unaffected; innocent.
  "Andy had a sweet, naive look when he smiled"

  Similar: innocent · unsophisticated · artless · ingenuous · inexperienced

- of or denoting art produced in a straightforward style that deliberately rejects sophisticated artistic techniques and has a bold directness resembling a child's work, typically in bright colors with little or no perspective.

Example: avocados, again

color	softness	variety	ripeness
bright green	firm	Zutano	unripe
green-black	medium	Hass	ripe
purple-black	firm	Hass	ripe
green-black	medium	Hass	unripe
purple-black	soft	Hass	ripe
bright green	firm	Zutano	unripe
green-black	soft	Zutano	ripe
purple-black	soft	Hass	ripe
green-black	soft	Zutano	ripe
green-black	firm	Hass	unripe
purple-black	medium	Hass	ripe

You have a soft green-black Hass avocado. Based on this data, would you predict that your avocado is ripe or unripe?

$$P(\text{ripe} \mid \text{soft, green-black, Hass}) = P(\text{ripe}) \cdot P(\text{soft} \mid \text{ripe}) \cdot P(\text{gb} \mid \text{ripe}) \cdot P(\text{hass} \mid \text{ripe})$$

$$7/11 \cdot 4/7 \cdot 3/7 \cdot 5/7$$

$$P(\text{unripe} \mid \text{soft, green-black, Hass}) = P(\text{unripe}) \cdot P(\text{soft} \mid \text{unripe}) \cdot P(\text{gb} \mid \text{unripe}) \cdot P(\text{hass} \mid \text{unripe})$$

$$4/11 \cdot 0$$

Uh oh...

- There are no soft unripe avocados in the data set.
- The estimate $P(\text{soft}|\text{unripe}) \approx \frac{\#\text{ soft unripe avocados}}{\#\text{ unripe avocados}}$ is 0.
- The estimated numerator, $P(\text{unripe}) \cdot P(\text{soft, green-black, Hass}|\text{unripe}) = P(\text{unripe}) \cdot P(\text{soft}|\text{unripe}) \cdot P(\text{green-black}|\text{unripe}) \cdot P(\text{Hass}|\text{unripe})$, is also 0.
- But just because there isn't a soft unripe avocado in the data set, doesn't mean that it's impossible for one to exist!
- **Idea:** Adjust the numerators and denominators of our estimate so that they're never 0.

Smoothing

- **Without** smoothing:

$$P(\text{soft}|\text{unripe}) \approx \frac{\#\text{ soft unripe}}{\#\text{ soft unripe} + \#\text{ medium unripe} + \#\text{ firm unripe}}$$

$$P(\text{medium}|\text{unripe}) \approx \frac{\#\text{ medium unripe}}{\#\text{ soft unripe} + \#\text{ medium unripe} + \#\text{ firm unripe}}$$

$$P(\text{firm}|\text{unripe}) \approx \frac{\#\text{ firm unripe}}{\#\text{ soft unripe} + \#\text{ medium unripe} + \#\text{ firm unripe}}$$

- **With** smoothing:

$$P(\text{soft}|\text{unripe}) \approx \frac{\#\text{ soft unripe} + 1}{\#\text{ soft unripe} + 1 + \#\text{ medium unripe} + 1 + \#\text{ firm unripe} + 1}$$

$$P(\text{medium}|\text{unripe}) \approx \frac{\#\text{ medium unripe} + 1}{\#\text{ soft unripe} + 1 + \#\text{ medium unripe} + 1 + \#\text{ firm unripe} + 1}$$

$$P(\text{firm}|\text{unripe}) \approx \frac{\#\text{ firm unripe} + 1}{\#\text{ soft unripe} + 1 + \#\text{ medium unripe} + 1 + \#\text{ firm unripe} + 1}$$

- When smoothing, we add 1 to the count of every group whenever we're estimating a conditional probability.

Example: avocados, with smoothing

color	softness	variety	ripeness
bright green	firm	Zutano	unripe
green-black	medium	Hass	ripe
purple-black	firm	Hass	ripe
green-black	medium	Hass	unripe
purple-black	soft	Hass	ripe
bright green	firm	Zutano	unripe
green-black	soft	Zutano	ripe
purple-black	soft	Hass	ripe
green-black	soft	Zutano	ripe
green-black	firm	Hass	unripe
purple-black	medium	Hass	ripe

You have a soft green-black Hass avocado. Using Naive Bayes, **with smoothing**, would you predict that your avocado is ripe or unripe?

Summary

Summary

- In classification, our goal is to predict a discrete category, called a **class**, given some features.
- The Naive Bayes classifier works by estimating the numerator of $P(\text{class}|\text{features})$ for all possible classes.
- It uses Bayes' theorem:

$$P(\text{class}|\text{features}) = \frac{P(\text{class}) \cdot P(\text{features}|\text{class})}{P(\text{features})}$$

- It also uses a simplifying assumption, that features are conditionally independent given a class:

$$P(\text{features}|\text{class}) = P(\text{feature}_1|\text{class}) \cdot P(\text{feature}_2|\text{class}) \cdot \ldots$$