Module 3 – Mean Squared Error and Empirical Risk Minimization

DSC 40A, Summer 2023

Agenda

- Recap from Module 2 – minimizing mean absolute error and formulating mean squared error.
- Minimizing mean squared error.
- Comparing different minimizers.
- Empirical risk minimization.

Recap from Module 2

The median minimizes mean absolute error

- Our problem was: find h^* which minimizes the mean absolute error, $R(h) = \frac{1}{n} \sum_{i=1}^{n} |y_i - h|$.

- **Regardless of if n is odd or even**, the answer is $h^* = \text{Median}(y_1, \dots, y_n)$. The **best prediction**, in terms of mean absolute error, is the **median**.
    - When n is odd, this answer is unique.
    - When n is even, any number between the middle two data points also minimizes mean absolute error.
    - We define the median of an even number of data points to be the mean of the middle two data points.

The mean absolute error is not differentiable

- We can't compute $\frac{d}{dh}|y_i - h|$.
- Remember: $|y_i - h|$ measures how far h is from y_i.
- Is there something besides $|y_i - h|$ which:
  1. Measures how far h is from y_i, *and*
  2. is **differentiable**?

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  2. is **differentiable**?

Discussion Question

Which of these would work?

a) $e^{|y_i - h|}$

b) $|y_i - h|^2$

c) $|y_i - h|^3$

d) $\cos(y_i - h)$

The squared error

- Let h be a prediction and y be the true value (i.e. the "right answer"). The **squared error** is:

$$|y - h|^2 = (y - h)^2$$

- Like absolute error, squared error measures how far h is from y.

- But unlike absolute error, the squared error is **differentiable**:

$$\frac{d}{dh}(y - h)^2 = \quad 2(h-y)$$

The new idea

- Find h^* by minimizing the **mean squared error**:

$$R_{\text{sq}}(h) = \frac{1}{n}\sum_{i=1}^{n}(y_i - h)^2$$

- Strategy: Take the derivative, set it equal to zero, and solve for the minimizer.

Minimizing mean squared error

$$R_{sq}(h) = \frac{1}{n}\sum_{i=1}^{n}(y_i - h)^2$$

$\frac{d}{dh} = 2(h-y)$

Discussion Question

Which of these is dR_{sq}/dh?

a) $\frac{1}{n}\sum_{i=1}^{n}(y_i - h)$

b) 0

c) $\sum_{i=1}^{n} y_i$

d) $\frac{2}{n}\sum_{i=1}^{n}(h - y_i)$

Solution

$$\frac{dR_{sq}}{dh} = \frac{d}{dh}\left[\frac{1}{n}\sum_{i=1}^{n}(y_i - h)^2\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\frac{d}{dh}(y_i - h)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}2(y_i - h)\cdot -1$$

$$\frac{2}{n}\sum_{i=1}^{n}(h - y_i)$$

Set to zero and solve for minimizer

$$\cancel{\frac{2}{n}} \sum_{i=1}^{n} (h - y_i) = 0$$

$$nh \quad \cancel{\sum_{i=1}^{n} h} - \sum_{i=1}^{n} y_i = 0$$

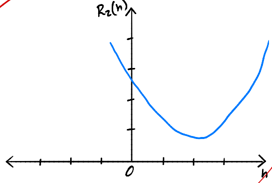
$$h = \underbrace{\frac{1}{n} \sum_{i=1}^{n} y_i}_{\star}$$

The mean minimizes mean squared error

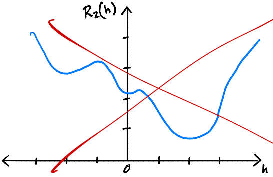
- Our new problem was: find h^* which minimizes the mean squared error, $R_{sq}(h) = \frac{1}{n}\sum_{i=1}^{n}(y_i - h)^2$.
  - The answer is: $\text{Mean}(y_1, \dots, y_n)$.
  - The **best prediction**, in terms of mean squared error, is the **mean**.
  - This answer is always unique!

Discussion Question

Suppose $y_1, \ldots, y_n$ are salaries. Which plot could be $R_{\text{sq}}(h)$?



(a)



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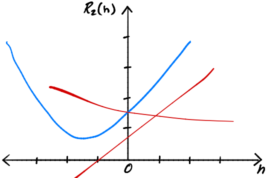
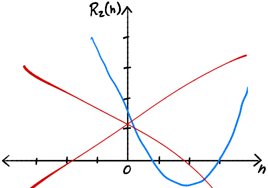


(c)



(d)

Comparing the median and mean

$O n$

easy to compute and parallelizable

$O(n \log n)$

$O n$ with quicksort

Outliers

- Consider our original dataset of 5 salaries.

  90,000 94,000 96,000 120,000 160,000

- As it stands, the **median is 96,000** and the **mean is 112,000**.

  +300,000

- What if we add 300,000 to the largest salary?

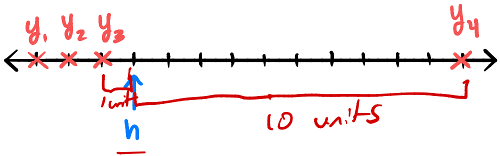
  90,000 94,000 96,000 120,000 460,000

- Now, the **median is still 96,000** but the **mean is 172,000**!

- **Key Idea:** The mean is quite **sensitive** to outliers.

Outliers

- The mean is quite **sensitive** to outliers.



- $|y_4 - h|$ is 10 times as big as $|y_3 - h|$.

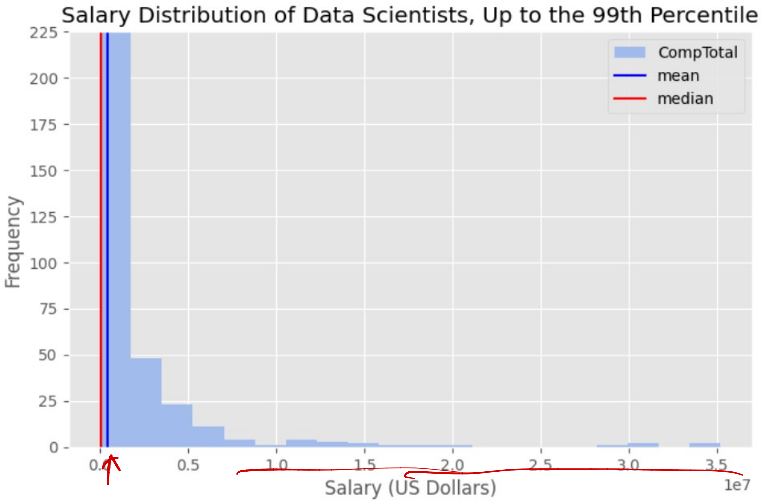
- But $(y_4 - h)^2$ is 100 times as big as $(y_3 - h)^2$.
  - This "pulls" h^* towards y_4.

- Squared error can be dominated by outliers.

Example: Data Scientist Salaries

- Dataset of 2,016 self-reported data science salaries in the United States from the 2022 StackOverflow survey.
- Median = \$86,700.
- Mean = \$501,425,531.
- Min = \$20.
- Max = \$1,000,000,000,000.
- 90th Percentile: \$700,000.

Example: Data Scientist Salaries



Example: Income Inequality

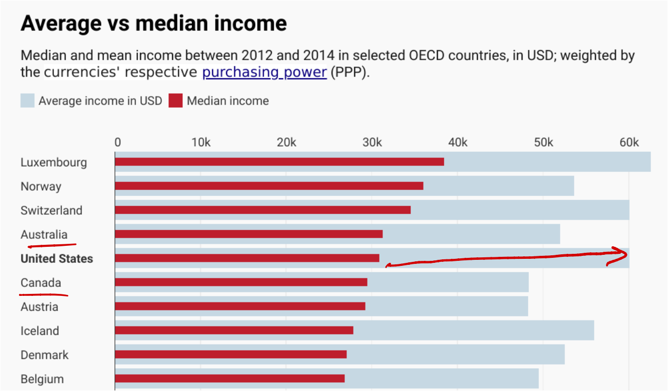
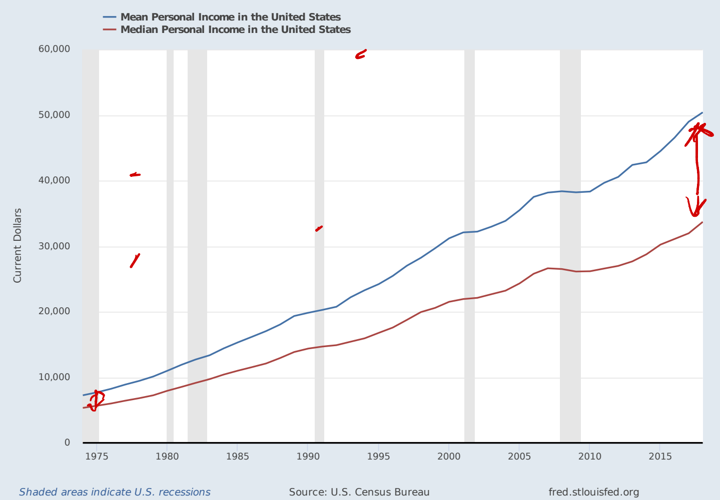


Chart: Lisa Charlotte Rost, Datawrapper

Example: Income Inequality



Empirical risk minimization

A general framework

- We started with the **mean absolute error**:

$$R(h) = \frac{1}{n}\sum_{i=1}^{n} |y_i - h|$$

- Then we introduced the **mean squared error**:

$$R_{\text{sq}}(h) = \frac{1}{n}\sum_{i=1}^{n} (y_i - h)^2$$

- They have the same form: both are averages of some measurement that represents how different h is from the data.

A general framework

- Definition: A **loss function** $L(h, y)$ takes in a prediction h and a true value (i.e. a "right answer"), y, and outputs a number measuring how far h is from y (bigger = further).

- The **absolute loss**:

regression

$$L_{\text{abs}}(h, y) = |y - h|$$

- The **squared loss**:

$$L_{\text{sq}}(h, y) = (y - h)^2$$

A general framework

- Suppose that $y_1, \dots, y_n$ are some data points, h is a prediction, and L is a loss function. The **empirical risk** is the average loss on the data set:

$$R_L(h) = \frac{1}{n} \sum_{i=1}^{n} \underbrace{L(h, y_i)}$$

- The goal of learning: find h that minimizes R_L. This is called **empirical risk minimization (ERM)**.

↳ "cost function"

The learning recipe

1. Pick a loss function. tons of options
2. Pick a way to minimize the average loss (i.e. empirical risk) on the data.

- **Key Idea**: The choice of loss function determines the properties of the result. **Different loss function = different minimizer = different prediction!**
  - Absolute loss yields the median.
  - Squared loss yields the mean.
  - The mean is easier to calculate but is more sensitive to outliers.

Example: 0-1 Loss

1. Pick as our loss function the **0-1 loss**:

$$L_{0,1}(h,y) = \begin{cases} 0, & \text{if } h = y \\ 1, & \text{if } h \neq y \end{cases}$$

2. Minimize empirical risk:

$$R_{0,1}(h) = \frac{1}{n}\sum_{i=1}^{n} L_{0,1}(h, y_i)$$

Example: 0-1 Loss

$y_1 \ldots y_n$ are distinct

$R_{0,1}(y_1)$

1. Pick as our loss function the **0-1 loss**:

$$L_{0,1}(h, y) = \begin{cases} 0, & \text{if } h = y \\ 1, & \text{if } h \neq y \end{cases}$$

$\frac{1}{n}\big(L_{0,1}(y_1, y_1)$ $\hookrightarrow 0$

$+ L_{0,1}(y_1, y_2)\big)$ $\hookrightarrow 1$

$\vdots$

$L_{0,1}(y_1, y_n)\big)$

2. Minimize empirical risk:

$$R_{0,1}(h) = \frac{1}{n}\sum_{i=1}^{n} L_{0,1}(h, y_i)$$

Discussion Question

Suppose $y_1, \ldots, y_n$ are all distinct. Find $R_{0,1}(y_1)$.

a) 0 b) $\frac{1}{n}$ c) $\frac{n-1}{n}$ d) 1

Minimizing empirical risk

example data set

$[-3, -2, -2, 2]$

$i=1 \quad i=2 \quad i=3 \quad i=4$

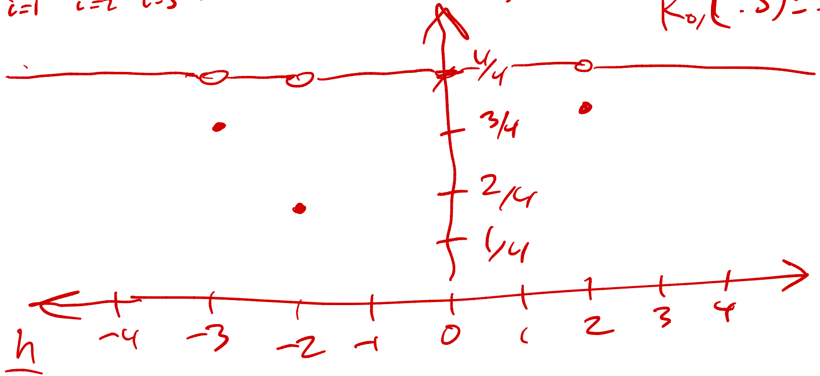
$$R_{0,1}(h) = \frac{1}{n} \sum_{i=1}^{n} \begin{cases} 0, & \text{if } h = y_i \\ 1, & \text{if } h \neq y_i \end{cases}$$

$R_{0,1}(-3) = \frac{1}{4} \cdot 3$

$R_{0,1}(-2) = \frac{1}{4} \cdot 2$

$R_{0,1}(2) = \frac{1}{4} \cdot 3$

$R_{0,1}(.5) = \frac{1}{4} \cdot 4$



Different loss functions lead to different predictions

common

Loss	Minimizer	Outliers	Differentiable
L_{abs}	median	**insensitive**	**no**
L_{sq}	mean	**sensitive**	**yes**
$L_{0,1}$	mode	**insensitive**	**no**

- The optimal predictions are all **summary statistics** that measure the **center** of the data set in different ways.

Summary

Summary

- $h^* = \text{Mean}(y_1, \dots, y_n)$ minimizes $R_{sq}(h) = \frac{1}{n} \sum_{i=1}^{n} (y_i - h)^2$, i.e. the mean minimizes mean squared error.
- The mean absolute error and the mean squared error fit into a general framework called **empirical risk minimization**.
  - Pick a loss function. We've seen absolute loss, $|y - h|^2$, squared loss, $(y - h)^2$, and 0-1 loss.
  - Pick a way to minimize the average loss (i.e. empirical risk) on the data.
- By changing the loss function, we change which prediction is considered the best.

Next time

- **Spread** – what is the meaning of the value of $R_{abs}(h^*)$? $R_{sq}(h^*)$?
- Creating a new loss function and trying to minimize the corresponding empirical risk.
  - We'll get stuck and have to look for a new way to minimize.