DSC 40A - Homework 1

due Friday, October 11th at 11:59PM

Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. Homeworks are due to Gradescope by 11:59PM on the due date.

Homework will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should **always explain and justify** your conclusions, using sound reasoning. Your goal should be to convince the reader of your assertions. If a question does not require explanation, it will be explicitly stated.

Homework should be written up and turned in by each student individually. You may talk to other students in the class about the problems and discuss solution strategies, but you should not share any written communication and you should not check answers with classmates. You can tell someone how to do a homework problem, but you cannot show them how to do it. We encourage you to type your solutions in LaTeX, using the Overleaf template on the course website.

For each problem you submit, you should **cite your sources** by including a list of names of other students with whom you discussed the problem. Instructors do not need to be cited.

This homework will be graded out of **51** points. The point value and difficulty of each problem or sub-problem is indicated by the number of avocados shown.

Note: For Problems 8(a) and 8(c), you'll need to code your answers in Python. More detailed instructions are provided in Problem 8. Note that to submit the homework, you'll have to submit your answers in PDF to the Homework 1 assignment on Gradescope, and submit your completed notebook `hw01-code.ipynb` to the Homework 1, Problems 8(a) and 8(c) autograder on Gradescope.

Brief explanation about proofs

In the following problems, you'll need to prove or disprove various statements about a dataset of numbers, $y_1, y_2, ..., y_n$. But first, let's discuss the general approach. (Note that this problem looks long, but most of it is us explaining *how* to answer it!

To prove that a statement is always **true**, you must provide some sort of reason as to why it is always true, no matter what the values $y_1, y_2, ..., y_n$ are. For example, consider the statement:

"Suppose we add 5 to each of $y_1, y_2, ..., y_n$. The mean of the new dataset must be greater than the mean of the original dataset."

This statement is always true, but it's not enough just to say "This statement is always true; since we're adding a positive number to each value, the mean will also increase." That's good intuition to have, but we need to provide a more rigorous justification. Here's what a more rigorous justification might look like:

"The mean of the original dataset is $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$. The mean of the new dataset is:

$$\frac{1}{n}\sum_{i=1}^{n}(y_i + 5) = \frac{1}{n}\left(\sum_{i=1}^{n} y_i + \sum_{i=1}^{n} 5\right) = \frac{1}{n}\left(\sum_{i=1}^{n} y_i + 5n\right) = \frac{1}{n}\sum_{i=1}^{n} y_i + 5 = \bar{y} + 5$$

Therefore, the mean of the new dataset is equal to the original dataset's mean plus 5, so the mean of the new dataset is greater than the mean of the original dataset, and so the statement is always true."

Note that in the argument above, we didn't assume anything specifically about the numbers in the original dataset — we didn't use a specific example. Just because a statement holds true for one example, doesn't mean it always holds true!

On the other hand, to disprove a statement, what you need to show is that it is **not** always true. The easiest way to do this is to provide a counterexample, i.e. a set of values $y_1, y_2, ..., y_n$ where the statement is false. For example, consider the statement:

"The smallest number in the dataset must be less than the mean."

Valid justification might look like:

"This statement is not always true. For example, consider the case where our dataset only contains one unique number, like $8, 8, 8$. Here, the mean is 8 and the smallest number is 8, so the smallest number is not less than the mean, and so the statement is not always true."

This is a counterexample, and is a sufficient disproof. (Fun fact: there exist entire books about counterexamples!)

Note that in both of the examples above, our answers clearly stated whether or not we thought the statement was always true. Your answers should do the same.

Now it's your turn!

Problem 1. Syllabus

Please confirm you have read the course syllabus.

Problem 2. Welcome Survey

Make sure to fill out the Welcome Survey, linked here for two points on this homework!

Problem 3. Reflection and Feedback Form

Make sure to fill out this Reflection and Feedback Form, linked here, for two points on this homework! This form is primarily for your benefit; research shows that reflecting and summarizing knowledge helps you understand and remember it.

Problem 4. Linear Functions, Mean and Median

Consider a dataset of numbers $y_1, y_2, ..., y_n$. Prove or find a counterexample to disprove each of the following statements.

Consider the linear function $f(x) = 9x - 4$.

a) If $a \leq b$, show that $f(a) \leq f(b)$.

b) Both of the statements below are true, but only one is a consequence of the property you proved in part (a). Which is it? Show that this statement is true, using the result of part (a).

1. $\text{Mean}(f(x_1), \ldots, f(x_n)) = f(\text{Mean}(x_1, \ldots, x_n))$
2. $\text{Median}(f(x_1), \ldots, f(x_n)) = f(\text{Median}(x_1, \ldots, x_n))$

c) Now, prove the other statement. Note your proof should not depend on the property you proved in part (a).

d) Suppose we consider a different linear function $g(x) = -2x + 5$. Prove or find a counterexample to disprove each of the following:

1. If $a \leq b$, then $g(a) \leq g(b)$.
2. $\text{Mean}(g(x_1), \ldots, g(x_n)) = g(\text{Mean}(x_1, \ldots, x_n))$

3. $\text{Median}(g(x_1), \ldots, g(x_n)) = g(\text{Median}(x_1, \ldots, x_n))$

Problem 5. Physics Experiment

On a sunny day, your friend Issac was daydreaming under an apple tree. All of a sudden, an apple dropped from the tree and hit Issac's head. From there Issac had the idea that the apple was dragged down from the tree by a "force". He contemplated an equation to describe this "force":

$$F = m * g$$

Where F is the force exerted onto the apple (in unit N), m is the mass of the apple (in unit kg), and g is a fundamental constant of nature, Issac named it the gravitational constant.

Issac was intrigued by this idea and he wanted to measure the gravitational constant. He designed a simple experiment to do this:

1. Collect 5 apples
2. Measure the mass of each of them and put them into a dataset $M = \{0.7, 0.7, 0.9, 0.8, 1.0\}$
3. Put each apple onto the tree branch, and let it drop from the tree
4. Measure the force exerted on each apple $F = \{6.9, 7.3, 9.8, 9.0, 10\}$

Issac carefully prepared his dataset so that there is one-to-one correspondence between each element in M and F. (i.e., the 1^{st} apple with measured mass 0.7 has an measured force 6.9, and the 3^{rd} apple with measured mass 0.9 has measured force of 9.8, etc.). Issac would like you to help him analyze these data to obtain the gravitational constant.

a) Calculate the Mean and Median for both Dataset M and F, make sure to show your work.

b) Given $\text{Mean}(F), \text{Mean}(M), \text{Median}(F), \text{Median}(M)$, Issac went ahead and reported:

$$\text{Mean}(g) = \frac{\text{Mean}(F)}{\text{Mean}(M)}$$
$$\text{Median}(g) = \frac{\text{Median}(F)}{\text{Median}(M)}$$

Issac's friend Albert does not agree with Issac on the reported values. Use the mathematical experssion of Mean(), to help Albert prove that Issac's report of $\text{Mean}(g)$ is incorrect.

State whether the following statements are True or False and give reasons for your answer:

c) Exactly half of the numbers in a data set must be smaller than the mean.

d) Not all of the numbers in the data set can be smaller than the mean.

Problem 6. An Alternative

In Lecture 2, we found that $h^* = \text{Mean}(y_1, y_2, ..., y_n)$ is the constant prediction that minimizes mean squared error:

$$R_{\text{sq}}(h) = \frac{1}{n}\sum_{i=1}^{n}(y_i - h)^2$$

To arrive at this result, we used calculus: we took the derivative of $R_{\text{sq}}(h)$ with respect to h, set it equal to 0, and solved for the resulting value of h, which we called h^*.

In this problem, we will minimize $R_{\text{sq}}(h)$ in a way that **doesn't** use calculus. The general idea is this: if $f(x) = (x-c)^2 + k$, then we know that f is a quadratic function that opens upwards with a vertex at (c, k), meaning that $x = c$ minimizes f. As we saw in class (see Lecture 2, slide 16), $R_{\text{sq}}(h)$ is a quadratic function of h!

Throughout this problem, let $y_1, y_2, ..., y_n$ be an arbitrary dataset, and let $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ be the mean of the y's.

a) 🥑🥑 What is the value of $\sum_{i=1}^{n}(y_i - \bar{y})$? Justify your answer.

b) 🥑🥑 Show that:

$$R_{\text{sq}}(h) = \frac{1}{n}\sum_{i=1}^{n}\left((y_i - \bar{y})^2 + 2(y_i - \bar{y})(\bar{y} - h) + (\bar{y} - h)^2\right)$$

Hint: To proceed, start by rewriting $y_i - h$ in the definition of $R_{sq}(h)$ as $(y_i - \bar{y}) + (\bar{y} - h)$. Why is this a valid step? Make sure not to expand unnecessarily.

c) 🥑🥑 Show that:

$$R_{\text{sq}}(h) = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 + (\bar{y} - h)^2$$

Hint: At some point, you will need to use your result from part (a).

d) 🥑 Why does the result in (c) prove that $h^* = \text{Mean}(y_1, y_2, ..., y_n)$ minimizes $R_{\text{sq}}(h)$?

Problem 7. Minimize risk or maximize likelihood?

In our lecture, we argued that one way to make a good prediction h is to minimize the mean absolute error:

$$R(h) = \frac{1}{n}\sum_{i=1}^{n}|h - x_i|.$$

We saw that the median of $y_1, \ldots, y_n$ is the prediction with the smallest mean error. Your friend Max thinks that instead of minimizing the mean error, it is better to maximize the following quantity:

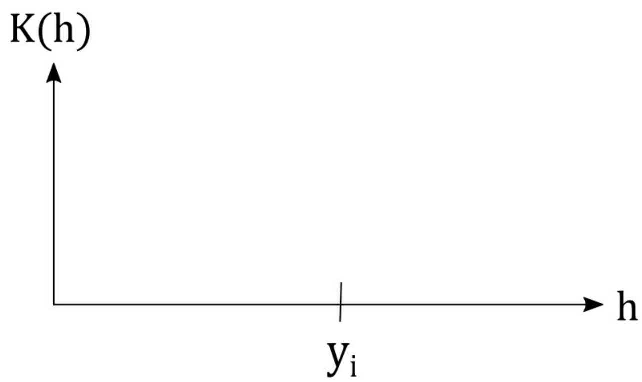
$$P(h) = \prod_{i=1}^{n} e^{-|h - y_i|}.$$

The above formula is written using product notation, which is similar to summation notation, except terms are multiplied and not added. For example,

$$\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdot \ldots \cdot a_n.$$

Max's reasoning is that for some models, $K(h) = e^{-|h - y_i|}$ is used to compute how likely prediction h will appear given the observation y_i – hence it is called "likelihood." Then, we should attempt to maximize the chance of getting the prediction h, given the set of observations. In this problem, we'll see if Max has a good idea.

a) 🥑🥑 On the axes below, sketch a graph of the basic shape of the likelihood function $K(h) = e^{-|h - y_i|}$. Label key points with their coordinates. Explain, based on the graph, why larger values of $K(h)$ correspond to better predictions h.



b) 🥑🥑🥑🥑 Recall that for a function of one variable $f(x)$, a value x^* is said to be a **minimizer** of $f(x)$ if

$$f(x^*) \leq f(x) \quad \text{for all } x.$$

Similarly, x^* is said to be a **maximizer** of $f(x)$ if

$$f(x^*) \geq f(x) \quad \text{for all } x.$$

Suppose that $f(x)$ is a function that is minimized at x^* and c is a positive constant. Show that the function $g(x) = e^{-c \cdot f(x)}$ is maximized at x^*.

c) 🥑🥑 At what value h^* is $P(h)$ maximized? Did Max have a reasonable idea?

Hint: Try writing out $P(h)$ by expanding the product notation.

Problem 8. Gaussian Location

Historical background (not necessary to solve the problem, but interesting):

Carl Friedrich Gauss was a German mathematician born in the 18th century, who is credited for several key ideas in mathematics that you're familiar with. For one, Gaussian elimination (also known as row reduction) in linear algebra is named after him. He's also credited for being the first person to minimize mean squared error — he did so to build a model to predict the locations of planets in the night sky.

Perhaps the most famous story involving Gauss is one from when he was just a child. His teacher, supposedly, asked him to add the integers from 1 through 100, expecting it to take him a while. However, within just a few seconds, he gave the answer 5050. In this problem, you'll use his insights to help you solve a DSC 20-style algorithmic problem efficiently.

The actual problem:

Here, we'll solve the "missing value problem." Suppose `vals` is a list of length $n - 1$, containing the integers from 1 to n, inclusive, with no duplicates and in unsorted order, but with exactly 1 value missing. Your job is to write a function that finds the missing integer in `vals`. For example, if `vals = [1, 2, 5, 3]`, the missing integer is `4`.

Parts (a) and (c) of this problem will require you to write code in this supplementary Jupyter Notebook. The code that you write in that notebook is autograded, both using public test cases that you can see in the notebook and hidden test cases that will only be run after you submit on Gradescope.

To submit your homework, in addition to submitting your answers PDF to the Homework 1 assignment on Gradescope, also submit `hw01-code.ipynb` to the Homework 1, Problems 6(a) and 6(c) autograder on Gradescope and wait until you see all public test cases pass!

a) In the linked supplementary notebook, complete the implementation of the function `missing_value_naive`. There's nothing you need to include in your answers PDF for this part.

b) Your implementation of `missing_value_naive` didn't need to be particularly efficient. To make our solution to the missing value problem more efficient, we'll use the fact that the missing value is the difference between the sum we'd expect if none of the values were missing, and the sum of the values we actually have. For example, if `vals = [1, 2, 5, 3]`, then $n = 5$, so the sum we'd expect is $1+2+3+4+5 = 15$, and so the missing value is $(1+2+3+4+5)-(1+2+5+3) = 15-11 = 4$.

To find the expected sum, instead of using a `for`-loop or something like `sum(range(1, n+1))`, we can turn to Gauss to make things even faster. Let's look at how he quickly derived that $1+2+...+99+100 = 5050$. Your job will then be to generalize what he did to $1 + 2 + ... + n$, and find a formula for the sum of the first n positive integers.

Let $S_{100} = 1 + 2 + ... + 99 + 100$. Gauss wrote out S_{100} in two different ways:

$$\begin{array}{llllllll} S_{100} = & 1 & + \; 2 & + \; 3 & + \; \cdots & + \; 98 & + \; 99 & + \; 100 \\ S_{100} = & 100 & + \; 99 & + \; 98 & + \; \cdots & + \; 3 & + \; 2 & + \; 1 \end{array}$$

Then, he noticed that each vertical pair of terms — 1 and 100, 2 and 99, 3 and 98, and so on, until 100 and 1 — each summed to 101. By adding the two lines above, he saw:

$$2S_{100} = 101 + 101 + 101 + ... + 101 + 101 + 101$$

Since there were 100 terms in each of the original equations for S_{100}, there were 100 terms equal to 101 in the equation above for $2S_{100}$. This let him solve for S_{100}:

$$\begin{aligned} 2S_{100} &= 101 + 101 + 101 + ... + 101 + 101 + 101 \\ 2S_{100} &= 100 \cdot 101 \\ S_{100} &= \frac{100 \cdot 101}{2} = 50 \cdot 101 = 5050 \end{aligned}$$

Now, it's your turn. Let $S_n = 1 + 2 + ... + n$. Find a closed-form expression for S_n, for any integer $n \geq 1$, and show your work. Your answer will be an arithmetic expression involving n; for example, an incorrect answer in the correct format could be $4n^3$.

Hint: This problem may seem daunting at first, but most of the work has already been done for you. What you need to do is repeat Gauss' work, but with an arbitrary integer n instead of 100. When you expand S_n twice, the way that Gauss expanded S_{100} twice, what is the sum of each vertical pair of terms? How many such terms are there? Verify that your answer is correct by testing it out on $1+2$, $1+2+3$, $1+2+3+4$, etc.

c) In the linked supplementary notebook, complete the implementation of the function `missing_value_fast`. There's nothing you need to include in your answers PDF for this part.

Problem 9.

Prove or disprove the following statements.

a) Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ be given vectors. If $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent, and $\{\mathbf{x}, \mathbf{z}\}$ is linearly independent, then $\mathbf{x}$ cannot be written as a linear combination of $\mathbf{y}$ and $\mathbf{z}$.

b) Let $\mathbf{x} \in \mathbb{R}^n$ and let $c \in \mathbb{R}$ be a nonzero scalar such that $\mathbf{x}^T(c\mathbf{x}) = 0$. Then $\mathbf{x}$ is the zero vector.

c) Let $A \in \mathbb{R}^{n\times n}$ be any matrix and then let

$$S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T A\mathbf{x} = 0\}.$$

Then S is a subspace of $\mathbb{R}^n$.