Lecture 3

Empirical Risk Minimization - mean absolute error

DSC 40A, Fall 2024

Announcements

- Groupwork 1 due Friday.

Agenda

- Recap: Mean squared error.
- Another loss function.
- Minimizing mean absolute error.

Question 🤔

Answer at q.dsc40a.com

Remember, you can always ask questions at q.dsc40a.com!

The modeling recipe

We've implicitly introduced a three-step process for finding optimal model parameters (like h^*) that we can use for making predictions:

1. Choose a model.

   $H(x) = h$

   Another choice: $H(x) = w_0 + w_1 x$

2. Choose a loss function.

   $L_{sq}(y_i, h) = (y_i - h)^2$

   Another choice?

3. Minimize average loss to find optimal model parameters.

   $h^* = \text{Mean}\{y_1, ..., y_n\}$

   Another optimal model h^*?

Recap: Mean squared error

Minimizing using calculus

We'd like to minimize:

$$\text{MSE} = R_{\text{sq}}(h) = \frac{1}{n}\sum_{i=1}^{n}\overbrace{(y_i - h)^2}^{L_{sq}(y_i, h)}$$

In order to minimize $R_{\text{sq}}(h)$, we:

1. take its derivative with respect to h,
2. set it equal to 0,
3. solve for the resulting h^*, and
4. perform a second derivative test to ensure we found a minimum.

The mean minimizes mean squared error!

- The problem we set out to solve was, find the h^* that minimizes:

$$R_{\text{sq}}(h) = \frac{1}{n}\sum_{i=1}^{n}(y_i - h)^2$$

- The answer is:

$$h^* = \text{Mean}(y_1, y_2, \ldots, y_n)$$

- The **best constant prediction**, in terms of mean squared error, is always the **mean**.
- This answer is always unique!
- We call h^* our **optimal model parameter**, for when we use:
  - the constant model, $H(x) = h$, and
  - the squared loss function, $L_{\text{sq}}(y_i, h) = (y_i - h)^2$.

Bonus: the mean is easy to compute

```
def mean(numbers):
    total = 0
    for number in numbers:
        total = total + number
    return total / len(numbers)
```

- Time complexity $\Theta(n)$

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Aside: Notation

Another way of writing

$$h^* \text{ is the value of } h \text{ that minimizes } \frac{1}{n}\sum_{i=1}^{n}(y_i - h)^2$$

is

$$h^* = \underset{h}{\operatorname{argmin}} \left(\frac{1}{n}\sum_{i=1}^{n}(y_i - h)^2 \right)$$

argument that minimizes

h^* is the solution to an **optimization problem**.

Another loss function

Another loss function

- Last lecture, we started by computing the **error** for each of our predictions, but ran into the issue that some errors were positive and some were negative.

$$e_i = y_i - H(x_i)$$

data we have

prediction $H(x) = h$

- The solution was to **square** the errors, so that all are non-negative. The resulting loss function is called **squared loss**.

$$L_{\text{sq}}(y_i, H(x_i)) = (y_i - H(x_i))^2$$

- Another loss function, which also measures how far $H(x_i)$ is from y_i, is **absolute loss**.

$$L_{\text{abs}}(y_i, H(x_i)) = |y_i - H(x_i)| \geq 0$$

80 minimized avg sqr loss
but not necessarily abs loss

Squared loss vs. absolute loss

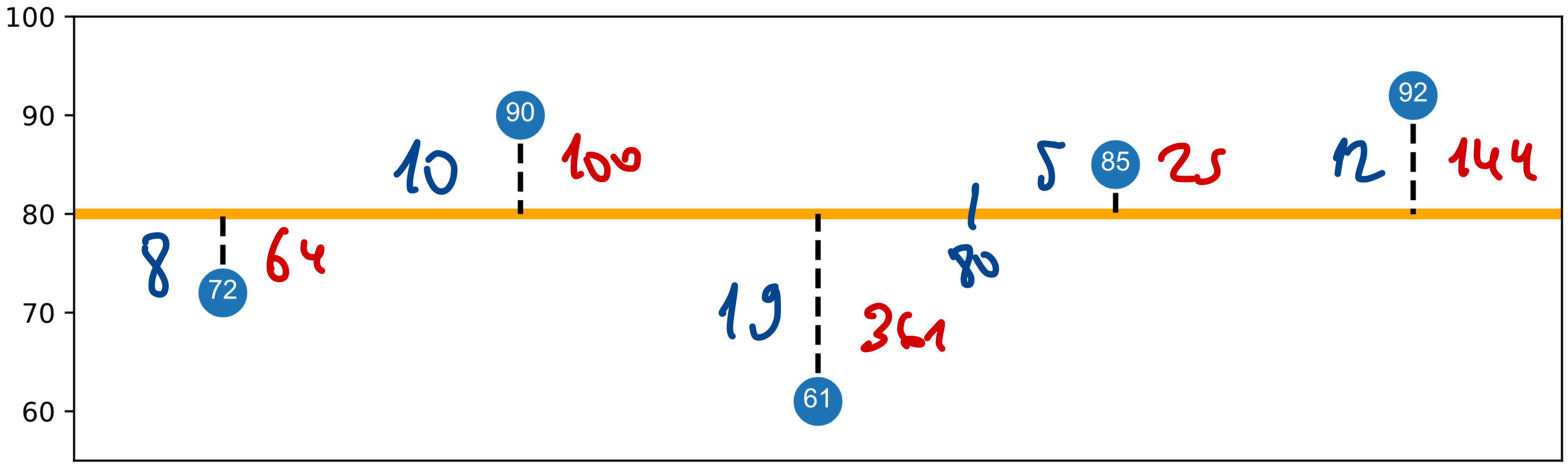
For the constant model, $H(x_i) = h$, so we can simplify our loss functions as follows:

- Squared loss: $L_{\text{sq}}(y_i, h) = (y_i - h)^2$.
- Absolute loss: $L_{\text{abs}}(y_i, h) = |y_i - h|$.

mean

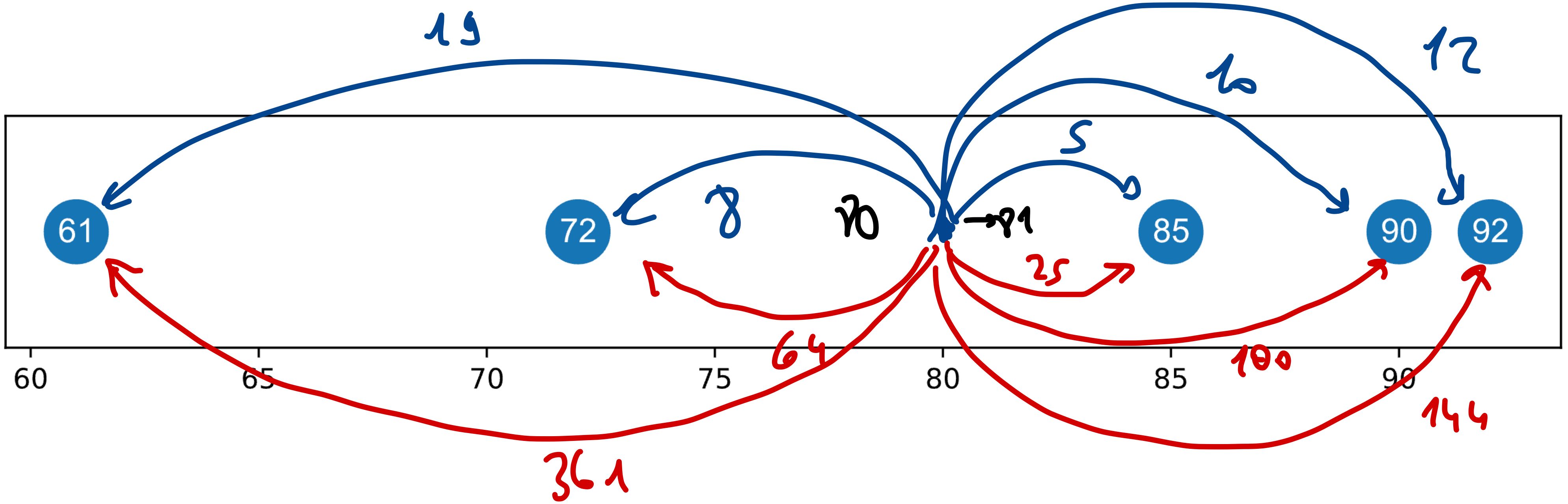
Consider, again, our example dataset of five commute times and the prediction $h = 80$.

$$y_1 = 72 \qquad y_2 = 90 \qquad y_3 = 61 \qquad y_4 = 85 \qquad y_5 = 92$$



Squared loss vs. absolute loss

- When we use squared loss, h^* is the point at which the average **squared** loss is minimized.
- When we use absolute loss, h^* is the point at which the average **absolute** loss is minimized.



Mean absolute error

- Suppose we collect n commute times, $y_1, y_2, \ldots, y_n$.
- The **average absolute loss**, or **mean absolute error (MAE)**, of the prediction h is:

$$R_{\text{abs}}(h) = \frac{1}{n} \sum_{i=1}^{n} \overbrace{|y_i - h|}^{L_{abs}(y_i, h)}$$

- We'd like to find the best prediction, h^*.
- Previously, when using squared loss we used calculus to find the optimal model parameter h^* that minimized R_{sq}.
- Can we use calculus to minimize $R_{\text{abs}}(h)$, too?

Minimizing mean absolute error

Minimizing using calculus, again

We'd like to minimize:

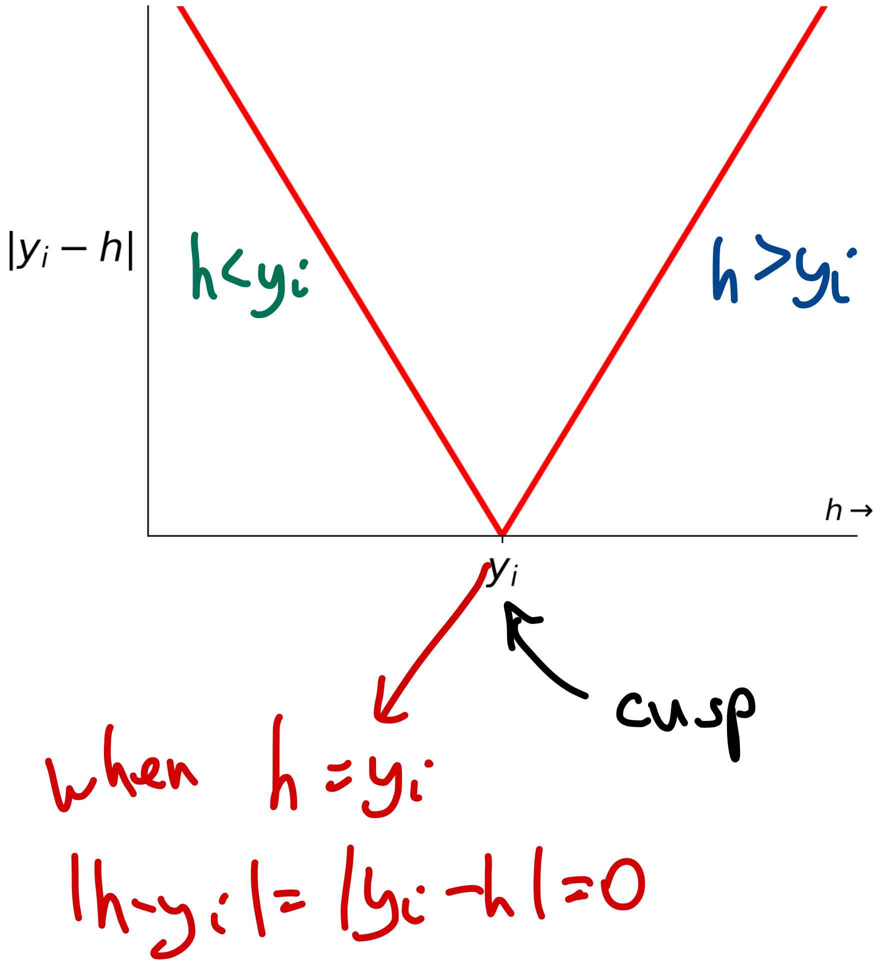
$$R_{\text{abs}}(h) = \frac{1}{n} \sum_{i=1}^{n} |y_i - h|$$

In order to minimize $R_{\text{abs}}(h)$, we:

1. take its derivative with respect to h,
2. set it equal to 0,
3. solve for the resulting h^*, and
4. perform a second derivative test to ensure we found a minimum.

find derivative of
$L_{abs}(y_i, h)$

Step 0: The derivative of $|y_i - h|$



Remember that $|x|$ is a **piecewise linear** function of x:

$$|x| = \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases}$$

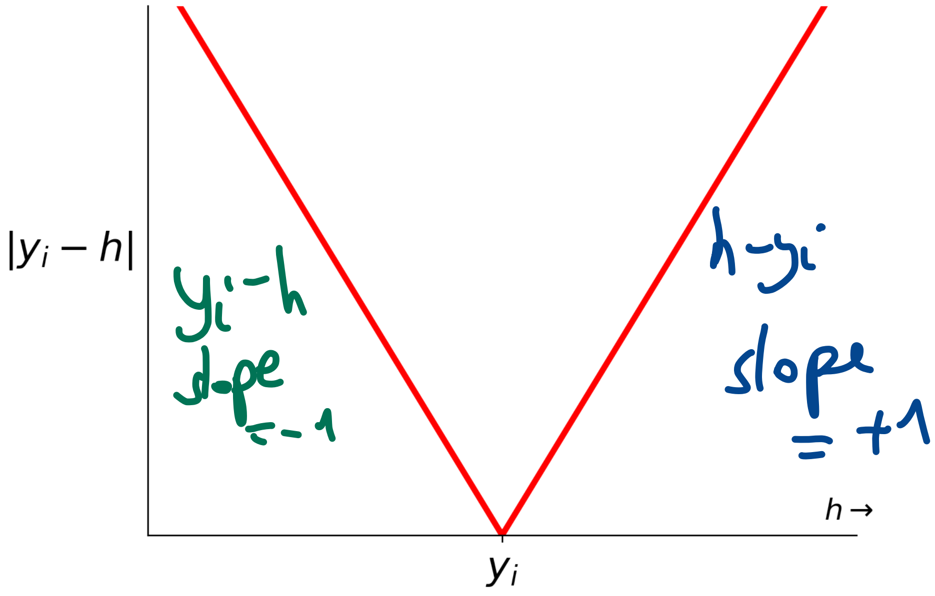
So, $|y_i - h|$ is also a piecewise linear function of h:

$$|y_i - h| = \begin{cases} y_i - h & h < y_i \\ 0 & y_i = h \\ h - y_i & h > y_i \end{cases}$$

if $y_i - h < 0$

$h - y_i > 0$

Step 0: The "derivative" of $|y_i - h|$



$$|y_i - h| = \begin{cases} y_i - h & h < y_i \\ 0 & y_i = h \\ h - y_i & h > y_i \end{cases}$$

What is $\frac{d}{dh}|y_i - h|$?

$$\frac{d}{dh}|y_i - h| = \begin{cases} -1 & , h < y_i \\ \text{undefined} & , h = y_i \\ +1 & , h > y_i \end{cases}$$

ignore for now!

Step 1: The "derivative" of $R_{\text{abs}}(h)$

$$\frac{d}{dh} |y_i - h| = \begin{cases} -1 & y_i > h \\ \text{undefined} & y_i = h \\ +1 & y_i < h \end{cases}$$

$$\frac{d}{dh} R_{\text{abs}}(h) = \frac{d}{dh}\left(\frac{1}{n}\sum_{i=1}^{n} |y_i - h|\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} \frac{d}{dh}|y_i - h|$$

this is a sum of $+1$'s and -1's

we add $+1$ when $h > y_i$

we subtract 1 when $h < y_i$

$$= \frac{1}{n}\left[\underbrace{\#(h > y_i)}_{+1} - \#(h < y_i)\right]$$

Example: 61, 72, 85, 90, 91

$h = 80$

$$\frac{d}{dh} R_{\text{abs}}(80) = \frac{2-3}{5} = -\frac{1}{5}$$

Question 🤔

Answer at q.dsc40a.com

The slope of R_{abs} at h is

$$\frac{1}{n}[(\# \text{ of } y_i < h) - (\# \text{ of } y_i > h)]$$

Suppose that the number of points n is odd. At what value of h does the slope change from negative to positive?

- A) h = mean of $\{y_1, \ldots, y_n\}$
- B) h = median of $\{y_1, \ldots, y_n\}$
- C) h = mode of $\{y_1, \ldots, y_n\}$

Steps 2 and 3: Set to 0 and solve for the minimizer, h^*

$$\frac{d}{dh} R_{abs}(h) = \frac{1}{n}\left[\#(h > y_i) - \#(h < y_i)\right] = 0$$

$$\#(h > y_i) = \#(h < y_i)$$

We want h^* is the value where the

number of data pts to left of h

=

number of data pts to right of h

$\Longrightarrow$ median!

The median minimizes mean absolute error!

- The new problem we set out to solve was, find the h^* that minimizes:

$$R_{\text{abs}}(h) = \frac{1}{n} \sum_{i=1}^{n} |y_i - h|$$

- The answer is:

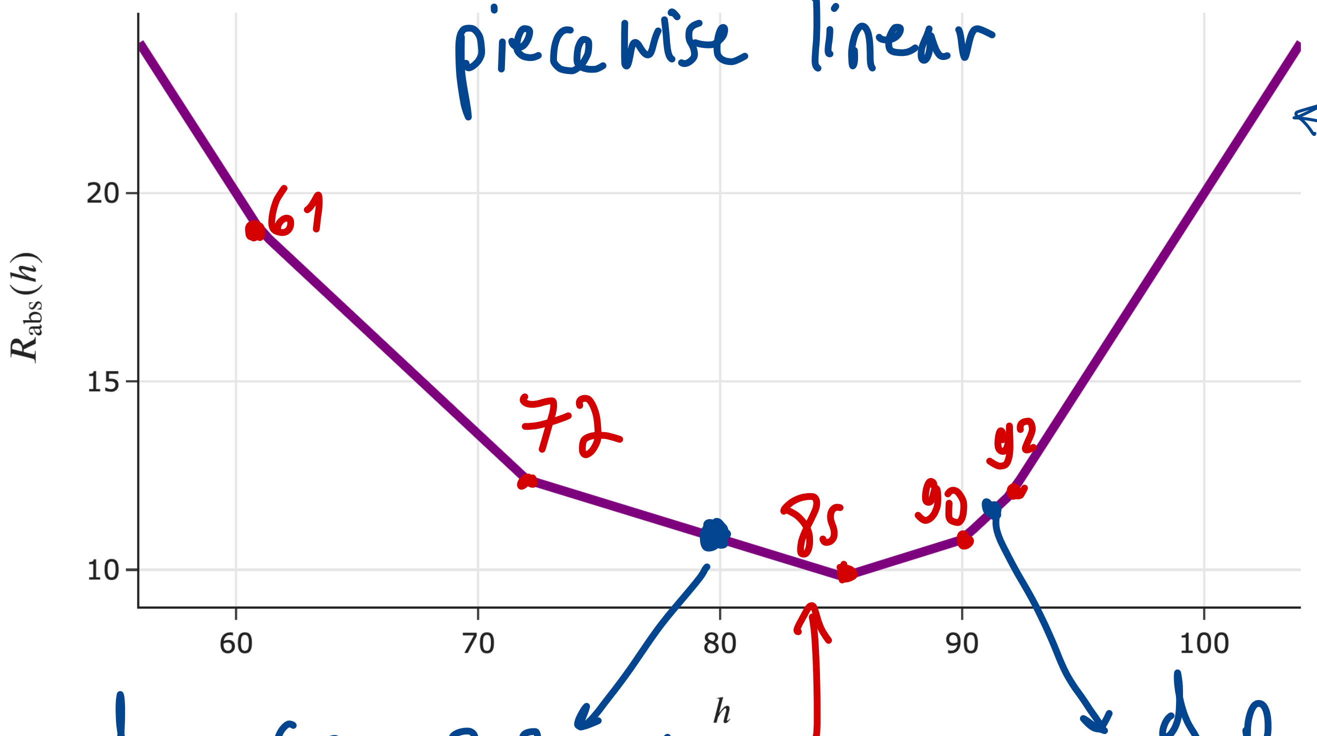
$$h^* = \text{Median}(y_1, y_2, \ldots, y_n)$$

- This is because the median has an equal number of data points to the left of it and to the right of it.
- To make a bit more sense of this result, let's graph $R_{\text{abs}}(h)$.

Visualizing mean absolute error

$= \frac{1}{5}\Big[$ 72 + 90 + 61 + 85 + 92 $\Big]$

$$R_{\text{abs}}(h) = \frac{1}{5}(|72 - h| + |90 - h| + |61 - h| + |85 - h| + |92 - h|)$$



$\frac{d}{dh} R_{abs}(80) = \frac{2-3}{5} = \frac{-1}{5}$

$h^* = \text{median}$

$\frac{d}{dh} R_{abs}(91) = \frac{4-1}{5} = \frac{3}{5}$

Consider, again, our example dataset of five commute times.

$$72, 90, 61, 85, 92$$

Where are the "bends" in the graph of $R_{\text{abs}}(h)$ – that is, where does its slope change?

Question 🤔

Answer at q.dsc40a.com

Consider, again, our example dataset of five commute times.

$$72, 90, 61, 85, 92$$

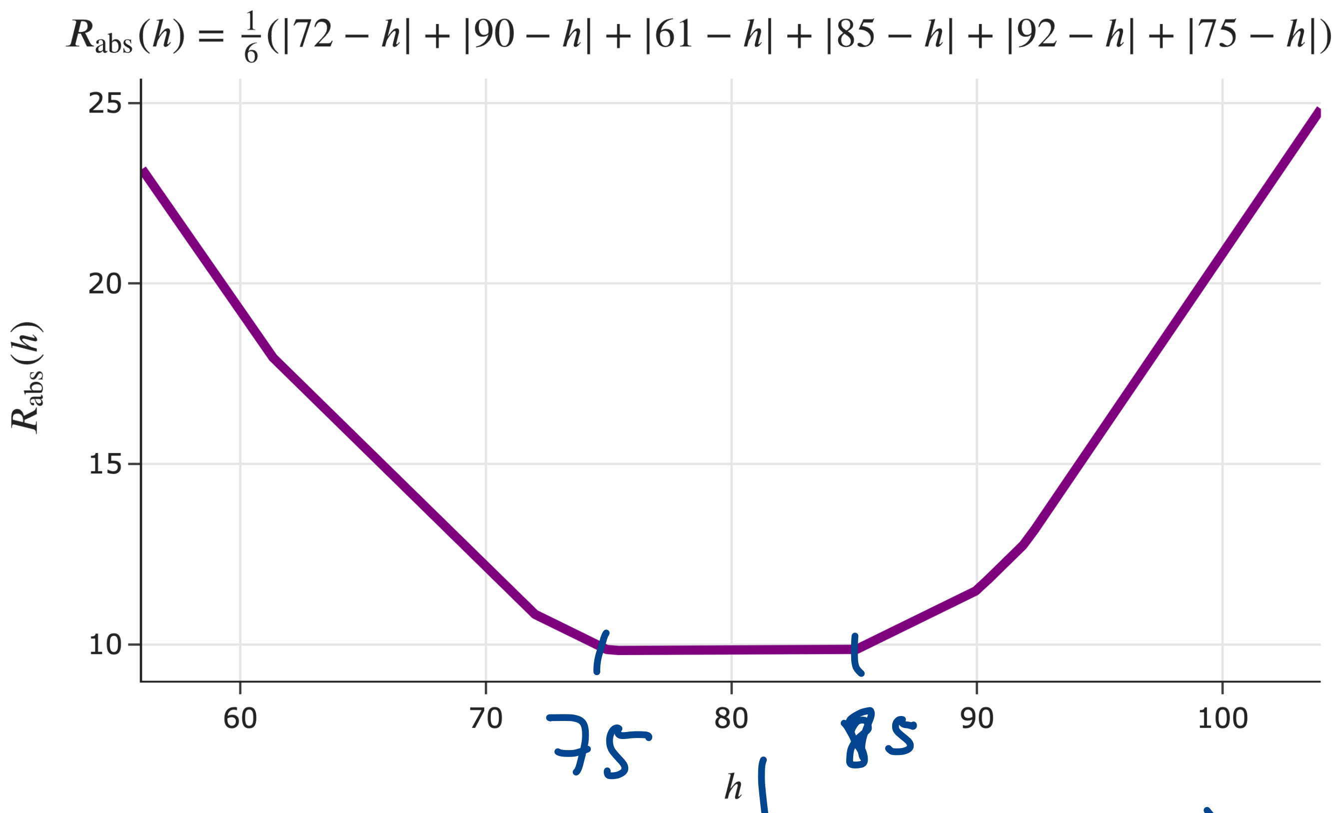
Suppose we add a sixth point so that our data is now

$$72, 90, 61, 85, 92, 75$$

Which of the following correctly describes the $h*$ that minimizes mean absolute error for our new dataset?

- A) 85 only
- B) 75 only
- C) 80 only
- D) Any value between 75 and 85 inclusive

Visualizing mean absolute error, with an even number of points



What if we add a sixth data point?

$$72, 90, 61, 85, 92, 75$$

Is there a unique h^*?

not unique if n is even

$R_{abs}(h) = R_{abs}(h^*)$ for all $75 \leq h \leq 85$

The median minimizes mean absolute error!

- The new problem we set out to solve was, find the h^* that minimizes:

$$R_{\text{abs}}(h) = \frac{1}{n} \sum_{i=1}^{n} |y_i - h|$$

- The answer is:

$$h^* = \text{Median}(y_1, y_2, \dots, y_n)$$

The **best constant prediction**, in terms of mean absolute error, is always the **median**.

- When n is odd, this answer is unique.
- When n is even, any number between the middle two data points (when sorted) also minimizes mean absolute error.
- When n is even, define the median to be the mean of the middle two data points.

The modeling recipe, again

We've now made two full passes through our "modeling recipe."

1. Choose a model.

2. Choose a loss function.

3. Minimize average loss to find optimal model parameters.

Empirical risk minimization

- The formal name for the process of minimizing average loss is **empirical risk minimization**.

- Another name for "average loss" is **empirical risk**.

- When we use the squared loss function, $L_{\text{sq}}(y_i, h) = (y_i - h)^2$, the corresponding empirical risk is mean squared error:

$$R_{\text{sq}}(h) = \frac{1}{n} \sum_{i=1}^{n} (y_i - h)^2$$

- When we use the absolute loss function, $L_{\text{abs}}(y_i, h) = |y_i - h|$, the corresponding empirical risk is mean absolute error:

$$R_{\text{abs}}(h) = \frac{1}{n} \sum_{i=1}^{n} |y_i - h|$$

Empirical risk minimization, in general

Key idea: If $L(y_i, h)$ is **any** loss function, the corresponding empirical risk is:

$$R(h) = \frac{1}{n} \sum_{i=1}^{n} L(y_i, h)$$

Question 🤔

Answer at q.dsc40a.com

What questions do you have?

Summary, next time

- $h^* = \text{Mean}(y_1, y_2, \ldots, y_n)$ minimizes mean squared error, $R_{\text{sq}}(h) = \frac{1}{n} \sum_{i=1}^{n} (y_i - h)^2$.
- $h^* = \text{Median}(y_1, y_2, \ldots, y_n)$ minimizes mean absolute error, $R_{\text{abs}}(h) = \frac{1}{n} \sum_{i=1}^{n} |y_i - h|$.
- $R_{\text{sq}}(h)$ and $R_{\text{abs}}(h)$ are examples of **empirical risk** – that is, average loss.
- **Next time**: What's the relationship between the mean and median? What is the significance of $R_{\text{sq}}(h^*)$ and $R_{\text{abs}}(h^*)$?

Shana tova to those celebrating!