Lecture 5

Simple Linear Regression

DSC 40A, Fall 2024

Announcements

- Homework 1 is due **Friday night**.
- Look at the office hours schedule here and plan to start regularly attending!
- Remember to take a look at the supplementary readings linked on the course website.

Agenda

- 0-1 loss
- Predictin rules using features
- Simple linear regression.
- Minimizing mean squared error for the simple linear model.

Question 🤔

Answer at q.dsc40a.com

Remember, you can always ask questions at q.dsc40a.com!

If the direct link doesn't work, click the "🤔 Lecture Questions" link in the top right corner of dsc40a.com.

Another example: 0-1 loss

Consider, for example, the **0-1 loss**:

$$L_{0,1}(y_i, h) = \begin{cases} 0 & y_i = h \\ 1 & y_i \neq h \end{cases}$$

The corresponding empirical risk is:

$$R_{0,1}(h) = \frac{1}{n} \sum_{i=1}^{n} L_{0,1}(y_i, h)$$

Question 🤔

Answer at q.dsc40a.com

$$R_{0,1}(h) = \frac{1}{n} \sum_{i=1}^{n} \begin{cases} 0 & y_i = h \\ 1 & y_i \neq h \end{cases}$$

Suppose $y_1, y_2, \ldots, y_n$ are all unique. What is $R_{0,1}(y_1)$?

- A. 0.
- B. $\frac{1}{n}$.
- C. $\frac{n-1}{n}$.
- D. 1.

Proportion of all points that aren't y_1

Minimizing empirical risk for 0-1 loss

$$R_{0,1}(h) = \frac{1}{n}\sum_{i=1}^{n}\begin{cases}0 & y_i = h\\ 1 & y_i \neq h\end{cases}$$

= proportion of all points not equal to h

1, 2, 3, 1, 4
mode is not unique

Minimized for $h = y_i$ where y_i is the most frequent value in the dataset.

$h^* = \text{mode}(y_1, y_2, \ldots, y_n)$

↑ most common value

Summary: Choosing a loss function

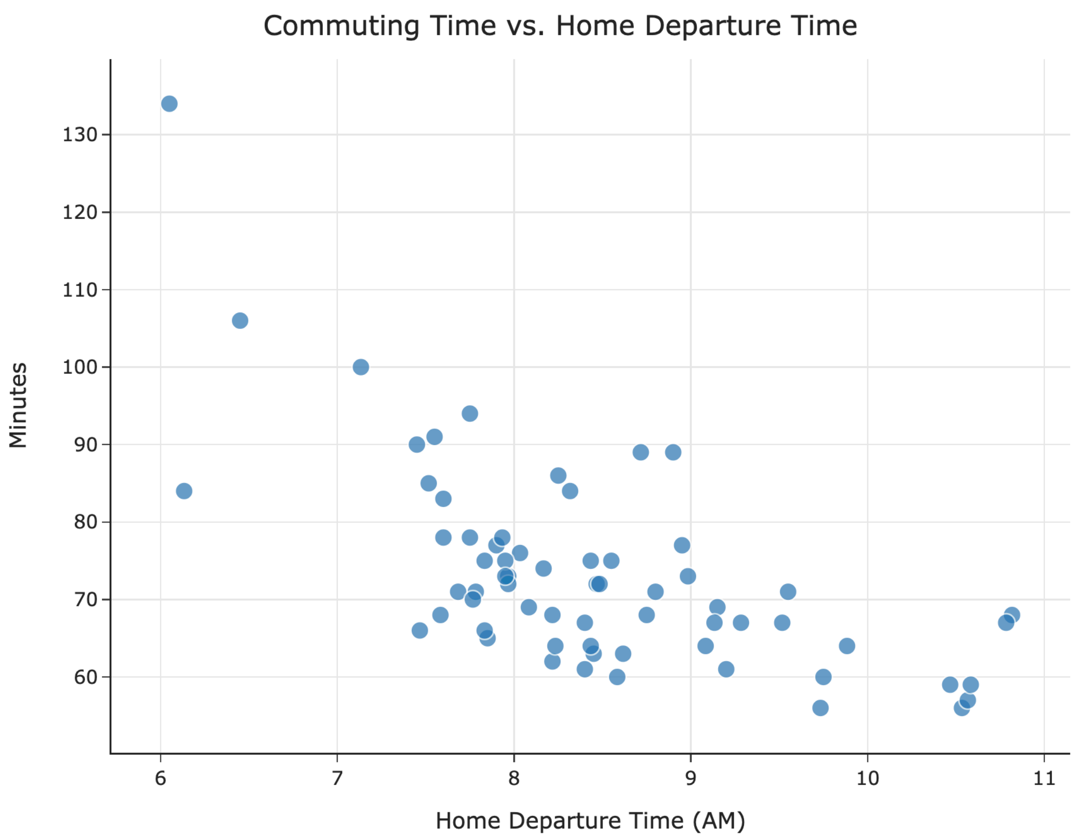
Key idea: Different loss functions lead to different best predictions, h^*!

Loss	Minimizer	Always Unique?	Robust to Outliers?	Differentiable?
L_{sq}	mean	yes ✅	no ❌	yes ✅
L_{abs}	median	no ❌	yes ✅	no ❌
L_∞	midrange	yes ✅	no ❌	no ❌
$L_{0,1}$	mode	no ❌	yes ✅	no ❌

The optimal predictions, h^*, are all **summary statistics** that measure the **center** of the dataset in different ways.

Predictions with features

Towards simple linear regression



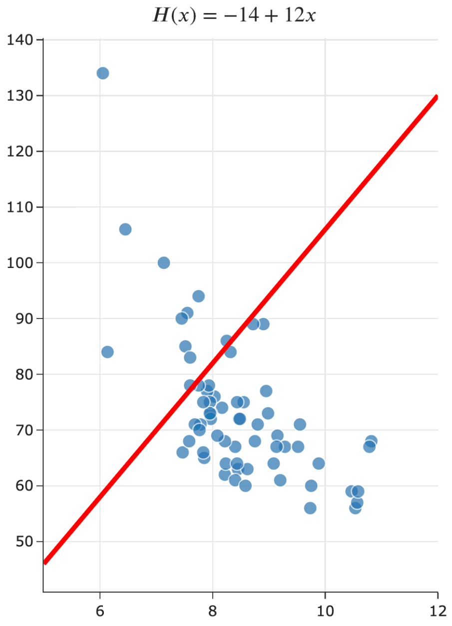
- In Lecture 1, we introduced the idea of a hypothesis function, $H(x)$.
- We've focused on finding the best **constant model**, $H(x) = h$.
- Now that we understand the modeling recipe, we can apply it to find the best **simple linear regression model**, $H(x) = w_0 + w_1 x$.
- This will allow us to make predictions that aren't all the same for every data point.

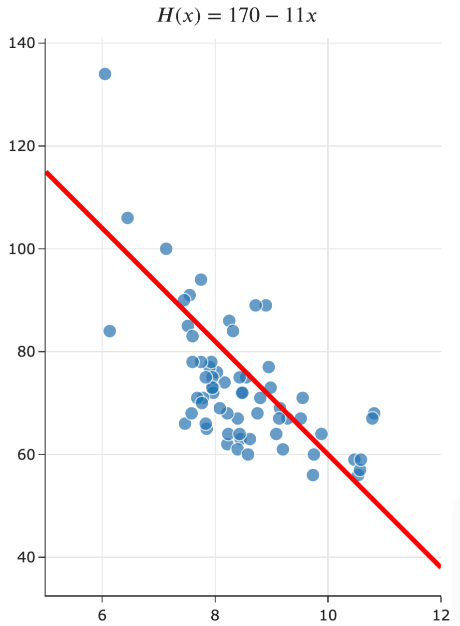
Recap: Hypothesis functions and parameters

A hypothesis function, H, takes in an x as input and returns a predicted y.

Parameters define the relationship between the input and output of a hypothesis function.

The simple linear regression model, $H(x) = w_0 + w_1 x$, has two parameters: w_0 and w_1.





The modeling recipe

1. Choose a model.

$H(x) = h$

constant

$H(x) =$ function of x

(not constant)

$= w_0 + w_1 x$

2. Choose a loss function.

$\ell(H(x_i), y_i)$ → abs → sqr → ...

3. Minimize average loss to find optimal model parameters.

find w_0, w_1

Features

A **feature** is an attribute of the data – a piece of information.

- **Numerical**: maximum allowed speed, time of departure
- **Categorical**: day of week
- **Boolean**: was there a car accident on the road?

Think of features as columns in a DataFrame (i.e. table).

x_i y_i

Departure time	Day of week	Accident on route	Commute time
7:05	Monday	yes	101
8:03	Tuesday	no	87
10:20	Wednesday	yes	79
8:30	Thursday	no	76

Modeling

- We believe that commute time is a function of departure time.
- I.e., there is a function H so that:

  commute time $\approx H$(departure time)

  y_i x_i
- H is called a **hypothesis function** or **prediction rule**.
- Our goal: find a good prediction rule H.

Possible Hypothesis Functions

- H_1(departure time) = 90 - 10 ·(departure time-7)
- H_2(departure time) = 90 - (departure time-8)^2
- H_3(departure time) = 20 + 6·departure time

These are all valid prediction rules.

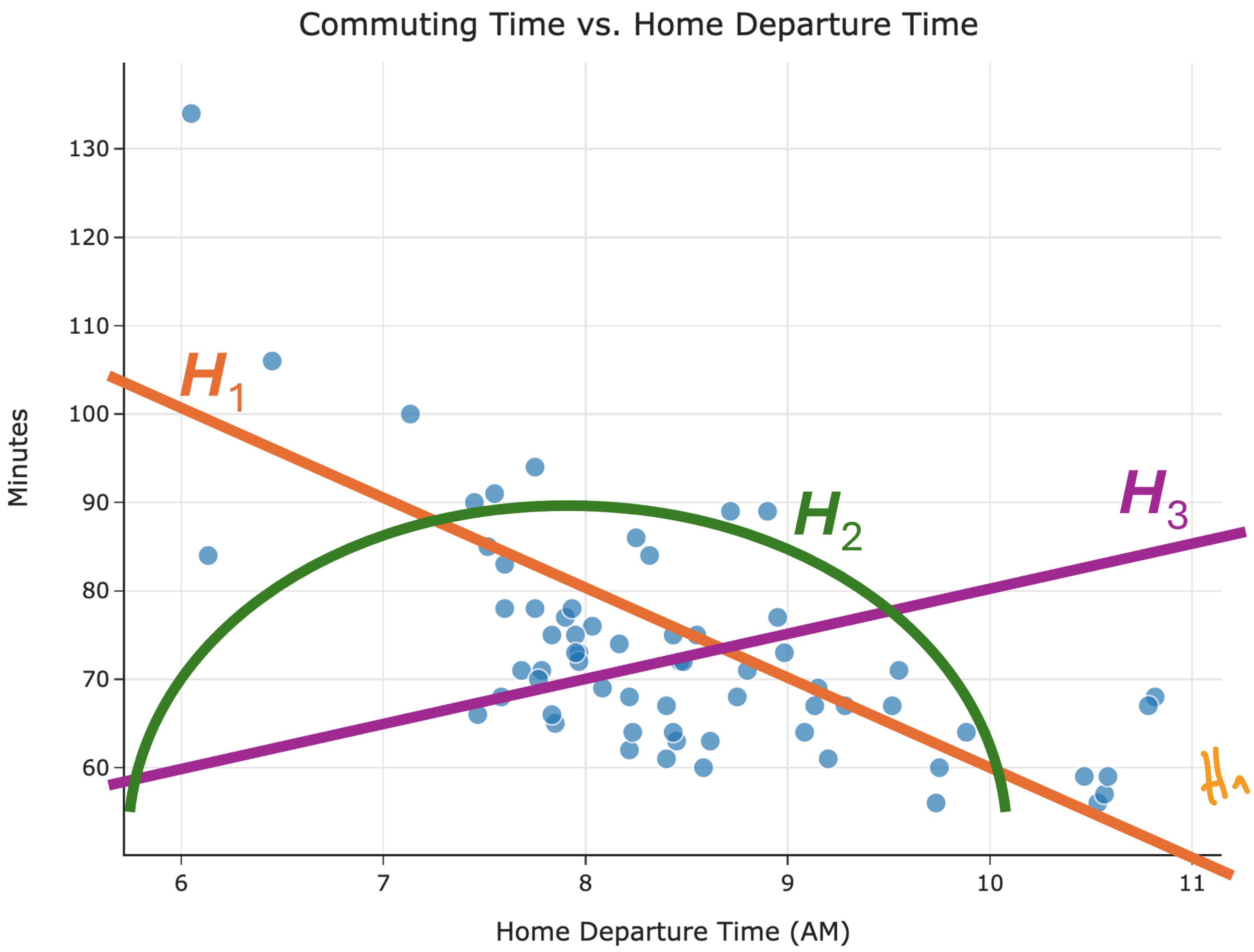
Some are better than others.

Comparing predictions

- How do we know which hyppthesis is best: $H_1,\ H_2,\ H_3$?
- We gather data from n days of commute. Let x_i be departure time, y_i be commute time:

  (departure time$_1$, commute time$_1$) $\quad (x_1, y_1)$
  (departure time$_2$, commute time$_2$) $\quad (x_2, y_2)$
  $\ldots \quad \rightarrow$
  (departure time$_n$, commute time$_n$) $\quad (x_n, y_n)$

- See which rule works better on data.



H_1 seems the best
but how do we know?

Quantifying the performance of a model

- Reminder: one loss function, which measures how far $H(x_i)$ is from y_i, is **absolute loss**. $|H(x_i) - y_i|$
- The mean absolute error of $H(x)$ is

$$R_{\text{abs}}(h) = \frac{1}{n} \sum_{i=1}^{n} |y_i - H(x_i))|$$

predicted commute

departure time

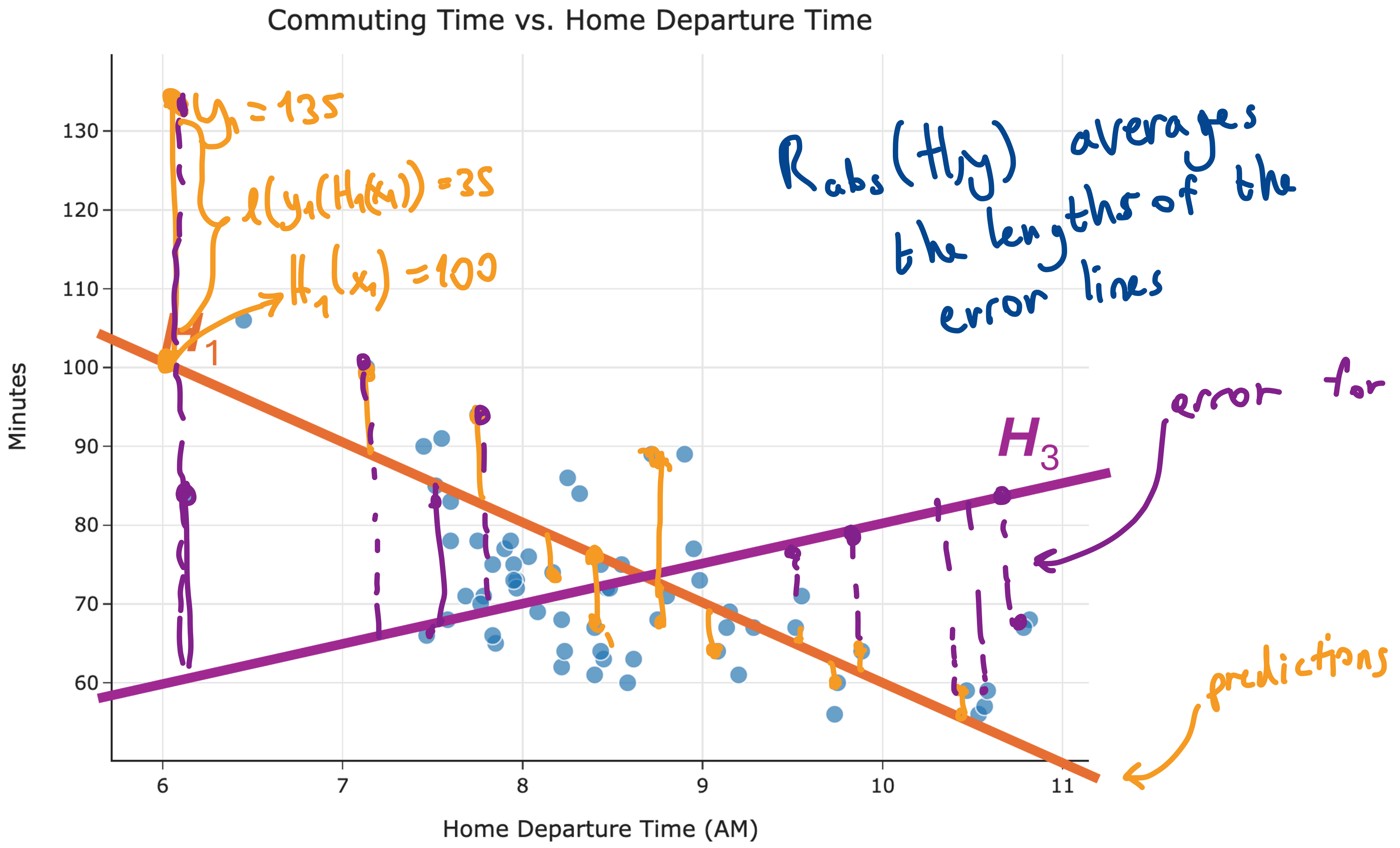
actual commute time

- We want the **best** prediction, $H^*(x)$.
- The smaller $R_{\text{abs}}(h)$ is, the better the hypothesis.

Mean absolute error

$R_{abs}(H_1, y) < R_{abs}(H_3, y)$

is worse than H_1



$R_{abs}(H, y)$ averages the lengths of the error lines

Finding the best hypothesis $H(x)$

- Goal: out of all functions$\mathbb{R} \to \mathbb{R}$, find the function H with the smallest mean absolute error.
- That is, H^* should be the function that minimizes

$$R_{\text{abs}}(h) = \frac{1}{n}\sum_{i=1}^{n} |y_i - H(x_i))|$$

Finding the best hypothesis $H(x)$

- Goal: out of all functions $\mathbb{R} \to \mathbb{R}$, find the function H with the smallest mean absolute error.
- That is, H^* should be the function that minimizes

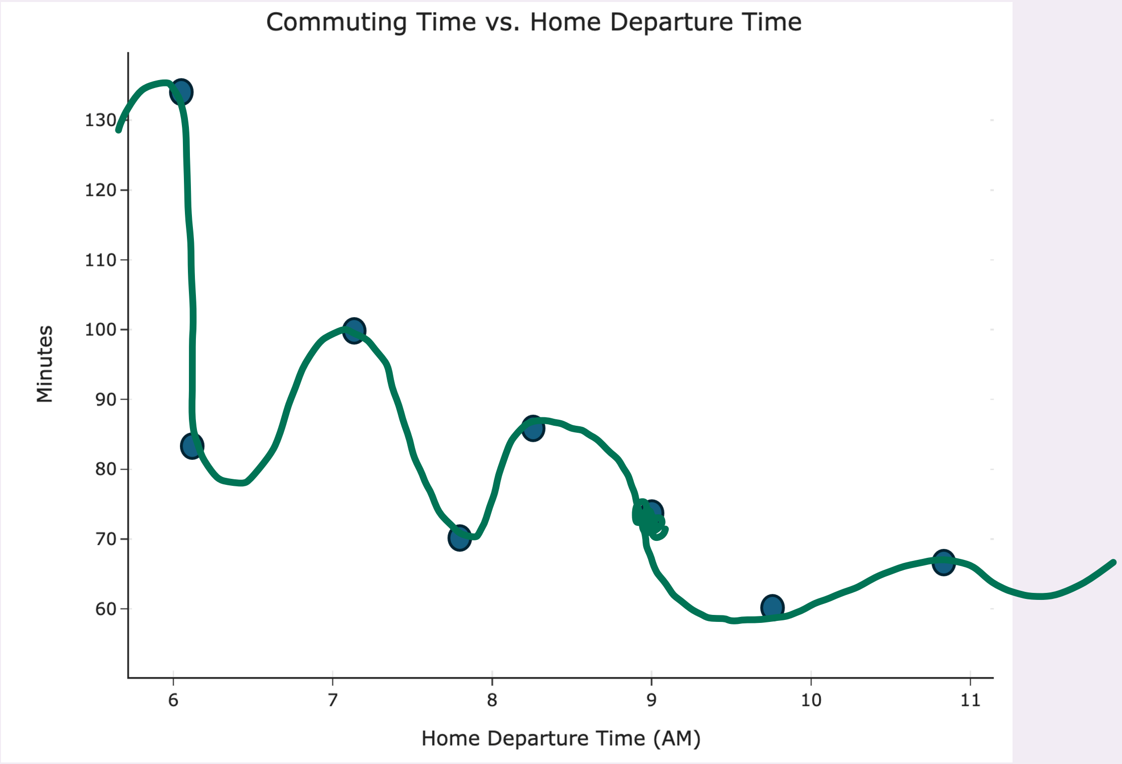
$$R_{\text{abs}}(h) = \frac{1}{n} \sum_{i=1}^{n} |y_i - H(x_i))|$$

- **There are two problems with this.**

Question 🤔 Answer at q.dsc40a.com

Given the data below, is there a prediction rule H which has zero mean absolute error?

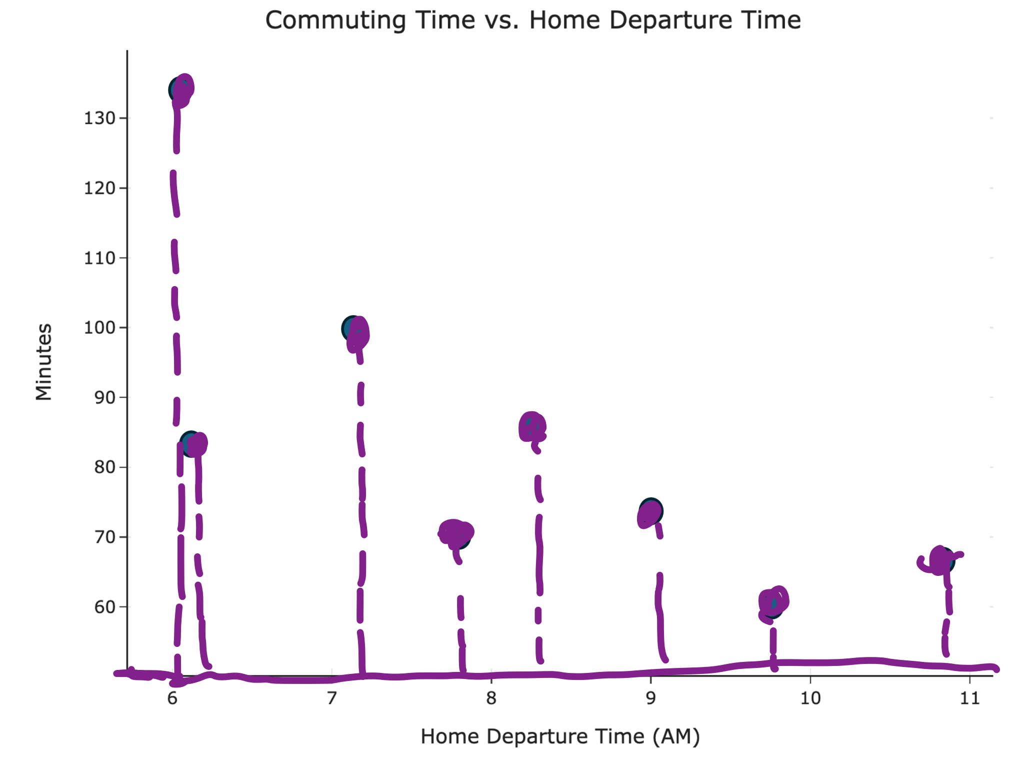
- A. yes
- B. no



Question 🤔 Answer at q.dsc40a.com

Given the data below, is there a prediction rule H which has zero mean absolute error?

- A. yes
- B. no



Problem

- We can make mean squared error very small, even zero!
- But the function will be weird.
- This is called **overfitting**.
- Remember our real goal: make good predictions on data **we haven't seen**.

Solution

- Don't allow H to be just any function.
- Require that it has a certain form.
- Examples:
  - Linear: $H(x) = w_0 + w_1 x$. ← this week
  - Quadratic: $H(x) = w_0 + w_1 x_1 + w_2 x^2$. ← In a few weeks
  - Exponential: $H(x) = w_0 e^{w_1 x}$.
  - Constant: $H(x) = w_0$. ← last week

non linear

Finding the best linear model

- **Goal:** Out of all **linear** functions $\mathbb{R} \to \mathbb{R}$, find the function H^* with the smallest mean squared error.
  - Linear functions are of the form $H(x) = w_0 + w_1 x$.
  - They are defined by a slope (w_1) and intercept (w_0).
- That is, H^* should be the linear function that minimizes

$$R_{abs}(H) = \frac{1}{n} \sum_{i=1}^{n} \left| y_i - H(x_i) \right|$$

Finding the best linear model

- **Goal:** Out of all **linear** functions $\mathbb{R} \to \mathbb{R}$, find the function H^* with the smallest mean squared error.
    - Linear functions are of the form $H(x) = w_0 + w_1 x$.
    - They are defined by a slope (w_1) and intercept (w_0).
- That is, H^* should be the linear function that minimizes

$$R_{abs}(H) = \frac{1}{n} \sum_{i=1}^{n} \left| y_i - H(x_i) \right|$$

-
    - There is still a problem with this.

Problem #2

It is hard to minimize the mean absolute error:

$$R_{abs}(H) = \frac{1}{n} \sum_{i=1}^{n} \left| y_i - H(x_i) \right|$$

- Not differentiable! we can't use calculus
- What can we do?

Minimizing mean squared error for the simple linear model

- We'll choose squared loss, since it's the easiest to minimize.
- Our goal, then, is to find the linear hypothesis function $H^*(x)$ that minimizes empirical risk:

MSE

$$R_{\text{sq}}(H) = \frac{1}{n} \sum_{i=1}^{n} \left(y_i - H(x_i)\right)^2$$

$\hookrightarrow H(x_i) = w_0 + w_1 x_i$

- Since linear hypothesis functions are of the form $H(x) = w_0 + w_1 x$, we can re-write R_{sq} as a function of w_0 and w_1:

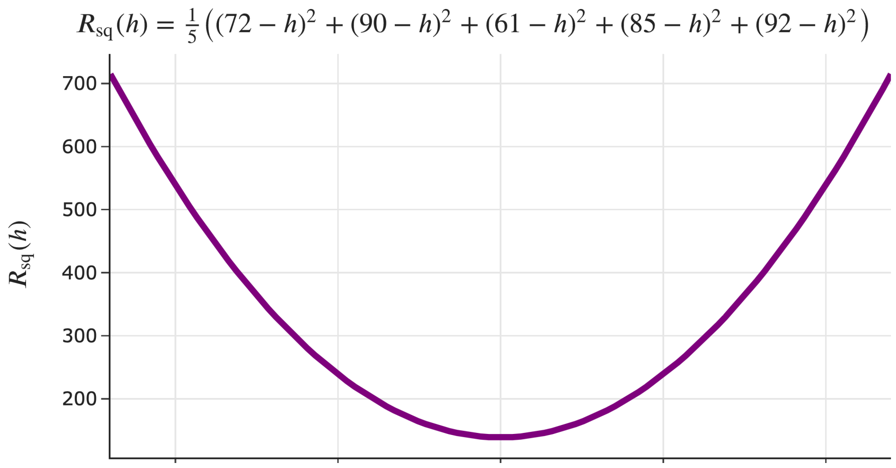
$$\boxed{R_{\text{sq}}(w_0, w_1) = \frac{1}{n} \sum_{i=1}^{n} \left(y_i - (w_0 + w_1 x_i)\right)^2}$$

same w_1 but not w_0

- **How do we find the parameters w_0^* and w_1^* that minimize $R_{\text{sq}}(w_0, w_1)$?**

Loss surface

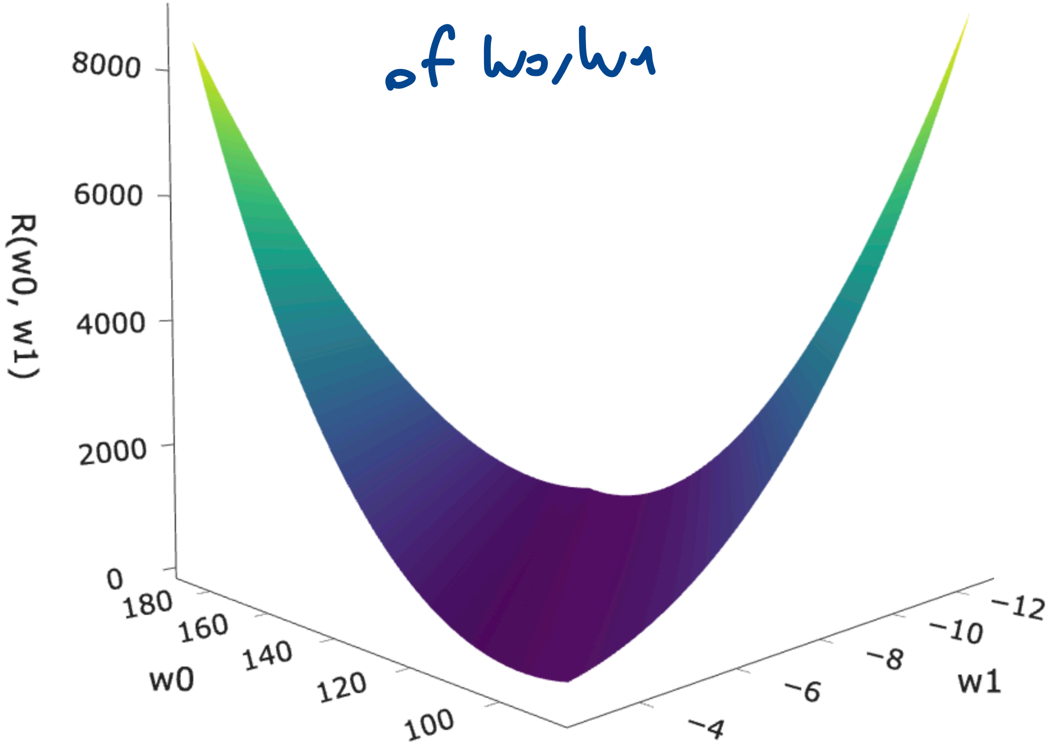
For the constant model, the graph of $R_{\text{sq}}(h)$ looked like a parabola.



single parameter h

What does the graph of $R_{\text{sq}}(w_0, w_1)$ look like for the simple linear regression model?

2d surface as func of w_0, w_1



Minimizing mean squared error for the simple linear model

Minimizing multivariate functions

- Our goal is to find the parameters w_0^* and w_1^* that minimize mean squared error:

$$R_{\text{sq}}(w_0, w_1) = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - (w_0 + w_1 x_i)\right)^2$$

- R_{sq} is a function of two variables: w_0 and w_1.
- To minimize a function of multiple variables:
  - Take partial derivatives with respect to each variable.
  - Set all partial derivatives to 0.
  - Solve the resulting system of equations.
  - Ensure that you've found a minimum, rather than a maximum or saddle point (using the second derivative test for multivariate functions).

Next time

$\frac{\partial R_{sq}}{\partial w_0}, \frac{\partial R_{sq}}{\partial w_1}$

w_0^*, w_1^*