Lecture 13 continued

Feature engineering and transformations

DSC 40A, Fall 2024

The Midterm Exam is on Monday, Nov 4th!

- Randomized seat assignment is in the homework - look up your seat.
- 50 minutes, on paper, no calculators or electronics.
    - **You are allowed to bring one two-sided page of notes.**
- Content: Lectures 1-13, Homeworks 1-4, Groupworks 1-4.
- Prepare by practicing with old exam problems at practice.dsc40a.com.
    - Problems are sorted by topic!

Discussion will take place today
HW 4 due Friday
No HW due next week

How do we fit hypothesis functions that aren't linear in the parameters?

- Suppose we want to fit the hypothesis function:

$$H(x) = w_0 e^{w_1 x}$$

- This is **not** linear in terms of w_0 and w_1, so our results for linear regression don't apply.
- **Possible solution**: Try to apply a **transformation**.

Goal: $w_0 + w_1 \cdot \square$

Transformations

- **Question**: Can we re-write $H(x) = w_0 e^{w_1 x}$ as a hypothesis function that **is** linear in the parameters?

$$y = w_0 e^{w_1 x}$$

/ $\log_e(\,)$

x	y	$z = \log(y)$

$$\log y = \log(w_0 e^{w_1 x})$$

$$= \log w_0 + \log e^{w_1 x}$$

$$\log y = \log w_0 + w_1 x$$

$$z = b_0 + b_1 x$$

linear in the parameters

$$T(x) \approx z$$

$$H(x) = e^z \approx y$$

$$b_0 = \log w_0 \Rightarrow w_0^* = e^{b_0}$$

Transformations

- **Solution**: Create a new hypothesis function, $T(x)$, with parameters b_0 and b_1, where $T(x) = b_0 + b_1 x$.
- This hypothesis function is related to $H(x)$ by the relationship $T(x) = \log H(x)$.
- $\vec{b}$ is related to $\vec{w}$ by $b_0 = \log w_0$ and $b_1 = w_1$.
- Our new observation vector, $\vec{z}$, is $\begin{bmatrix} \log y_1 \\ \log y_2 \\ \ldots \\ \log y_n \end{bmatrix}$.

$$X^T X \vec{b}^* = X^T \vec{z}$$

new observation vector

intermediate parameter vector

- $T(x) = b_0 + b_1 x$ is linear in its parameters, b_0 and b_1.
- Use the solution to the normal equations to find $\vec{b}^*$, and the relationship between $\vec{b}$ and $\vec{w}$ to find $\vec{w}^*$.

Once again, let's try it out! Follow along in this notebook.

Non-linear hypothesis functions in general

- Sometimes, it's just not possible to transform a hypothesis function to be linear in terms of some parameters.
- In those cases, you'd have to resort to other methods of finding the optimal parameters.
    - For example, $H(x) = w_0 \sin(w_1 x)$ **can't** be transformed to be linear.
    - But, there are other methods of minimizing mean squared error:

$$R_{\text{sq}}(w_0, w_1) = \frac{1}{n} \sum_{i=1}^{n} (y_i - w_0 \sin(w_1 x))^2$$

    - One method: **gradient descent**, the topic of the next lecture!
- Hypothesis functions that are linear in the parameters are much easier to work with.

Question 🤔

Answer at q.dsc40a.com

Which hypothesis function is **not** linear in the parameters?

- A. $H(\vec{x}) = w_1 (x^{(1)} x^{(2)}) + \frac{w_2}{x^{(1)}} \sin\left(x^{(2)}\right)$
- B. $H(\vec{x}) = 2^{w_1} x^{(1)}$
- C. $H(\vec{x}) = \vec{w} \cdot \text{Aug}(\vec{x})$
- D. $H(\vec{x}) = w_1 \cos(x^{(1)}) + w_2 2^{x^{(2)} \log x^{(3)}}$
- E. More than one of the above.

We want $H(x) = \sum_{i=0}^{d} w_i \square$ ← no w's

$= w_1 \square + w_2 \square$

$\frac{\sin(x^{(2)})}{x^{(1)}}$

$= w_0 + w_1 x^{(1)} + w_2 x^{(2)} + \dots$

we can apply transformation

Roadmap

- This is the end of the content that's in scope for the Midterm Exam.
- Now, we'll introduce **gradient descent**, a technique for minimizing functions that can't be minimized directly using calculus or linear algebra.
- After the Midterm Exam, we'll:
  - Switch gears to **probability**.

Lecture 14

Gradient Descent

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Agenda

- Minimizing functions using gradient descent.
- Convexity.
- More examples.
    - Huber loss.
    - Gradient descent with multiple variables.

Question 🤔

Answer at q.dsc40a.com

Remember, you can always ask questions at q.dsc40a.com!

If the direct link doesn't work, click the "🤔 Lecture Questions" link in the top right corner of dsc40a.com.

The modeling recipe

1. Choose a model.

① $H(x) = h$ constant

② $H(x) = w_0 + w_1 x$

simple linear regression

③ $H(\vec{x}) = w_0 + w_1 x^{(1)} + w_2 x^{(2)} + \ldots + w_d x^{(d)}$

$= \vec{w} \cdot \text{Aug}(\vec{x})$

$\vec{h} = X\vec{w}$

2. Choose a loss function.

Ⓐ squared loss $(y_i - H(x_i))^2$

Ⓑ absolute loss $|y_i - H(x_i)|$

Ⓒ 0-1 loss

3. Minimize average loss to find optimal model parameters.

(average loss = empirical risk)

①Ⓐ $R_{sq}(h) = \frac{1}{n} \sum_i (y_i - h)^2 \Rightarrow h^* = \text{mean}\{y_1, \ldots, y_n\}$

③Ⓐ $R_{sq}(\vec{w}) = \frac{1}{n} \|\vec{y} - X\vec{w}\|^2 = \|\vec{e}\|^2$

Minimizing functions using gradient descent

Minimizing empirical risk

- Repeatedly, we've been tasked with **minimizing** the value of empirical risk functions.
  - Why? To help us find the **best** model parameters, h^* or w^*, which help us make the **best** predictions!
- We've minimized empirical risk functions in various ways.
  - $R_{\text{sq}}(h) = \frac{1}{n} \sum_{i=1}^{n} (y_i - h)^2$ calculus
  - $R_{\text{abs}}(w_0, w_1) = \frac{1}{n} \sum_{i=1}^{n} |y_i - (w_0 + w_1 x)|$ python
  - $R_{\text{sq}}(\vec{w}) = \frac{1}{n} \|\vec{y} - X\vec{w}\|^2$ linear algebra

Minimizing arbitrary functions

- Assume $f(t)$ is some **differentiable** single-variable function.
- When tasked with minimizing $f(t)$, our general strategy has been to:
  i. Find $\frac{df}{dt}(t)$, the derivative of f.
  ii. Find the input t^* such that $\frac{df}{dt}(t^*) = 0$.

can't isolate t!

$t^* =$

- However, there are cases where we can find $\frac{df}{dt}(t)$, but **it is either difficult or impossible to solve** $\frac{df}{dt}(t^*) = 0$.

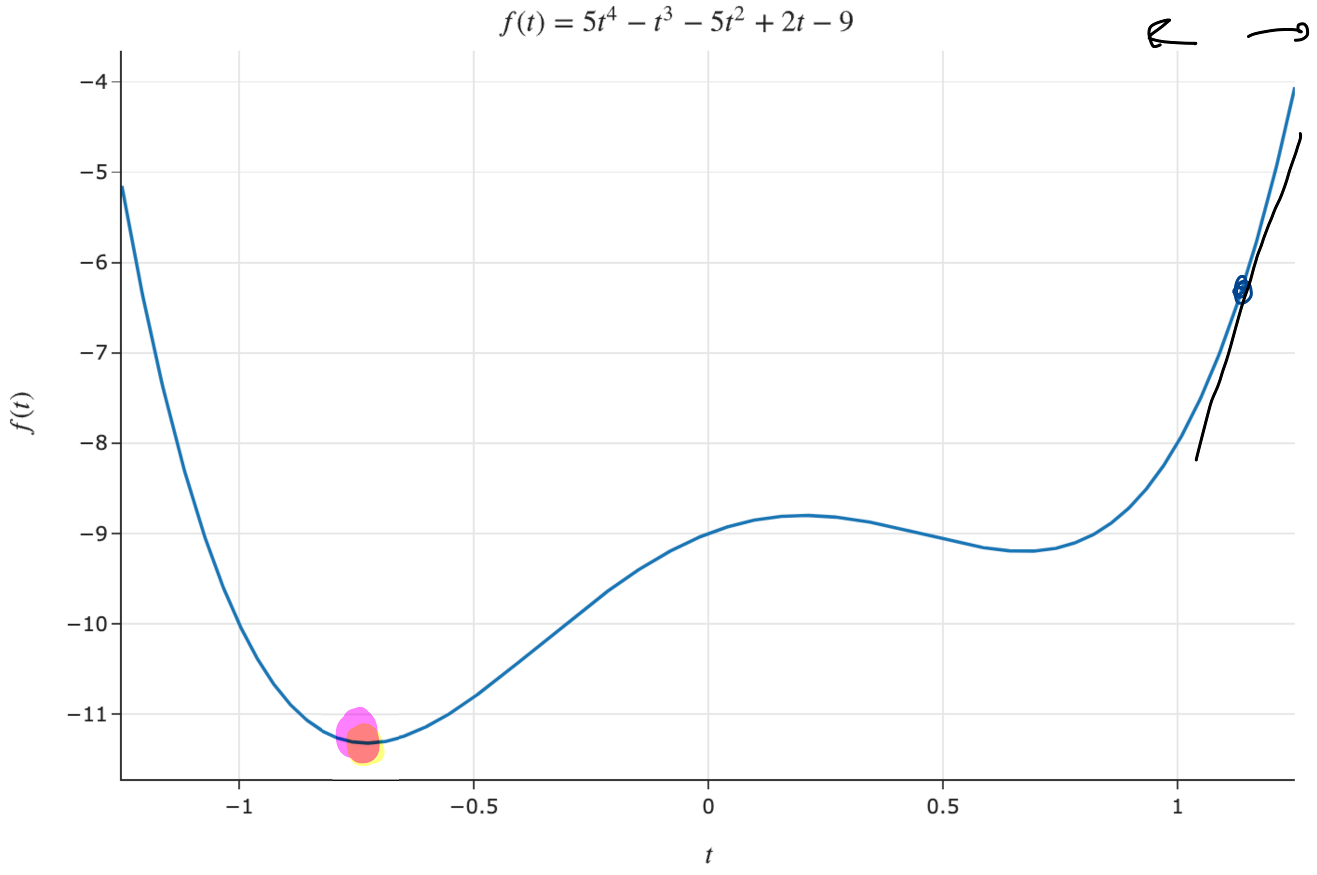
$$f(t) = 5t^4 - t^3 - 5t^2 + 2t - 9$$

$$\frac{df(t)}{dt} = 20t^3 - 3t^2 - 10t + 2t$$

- Then what?

What does the derivative of a function tell us?

- **Goal**: Given a **differentiable** function $f(t)$, find the input t^* that minimizes $f(t)$.
- What does $\frac{d}{dt} f(t)$ mean?



Let's go hiking!

- Suppose you're at the top of a mountain 🏔️ and need to get **to the bottom**.
- Further, suppose it's really cloudy ☁️, meaning you can only see a few feet around you.
- **How** would you get to the bottom?